



Revenue in Resource Allocation Games and Applications

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REVENUE IN RESOURCE ALLOCATION GAMES AND APPLICATIONS

A Dissertation

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REVENUE IN RESOURCE ALLOCATION GAMES AND APPLICATIONS

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This dissertation studies a general class of resource allocation games in computer systems. The applications of these games include sharing network bandwidth, scheduling jobs in data centers and distributing click-through resources in sponsored search.

The main focus of the dissertation is the *revenue* that can be obtained by providers. We investigate the revenue of proportional sharing under a symmetry condition among users, and show how to modify this mechanism to get a competitive revenue without the symmetry condition. We study the weighted proportional sharing mechanism as a natural extension of fair sharing to capture the incentives of revenue maximizing providers.

BIOGRAPHICAL SKETCH

Thành was born on 28 December, 1979 in *làng Bùng*, a small village located in the Red River Delta of the northern part of Vietnam. Làng Bùng is a remote village in Bac Ninh province, they did not have electricity until the late eighties. Thành's mother was one of the few teachers in the village, most other people were farmers growing rice and sweet potatoes.

At the age of fourteen Thành was sent to Hanoi, the capital city, for a better school. That was the beginning of his journey away from his family. At the age of eighteen Thành traveled to Budapest to study at Eötvös University. He studied Hungarian and mathematics in Budapest during academic years and worked at Lake Balaton in summers. After six years in Hungary, Thành continued his journey to the United States. Since then, he has been living in Ithaca, NY, doing research in applied mathematics at Cornell University.

When writing these lines Thành is close to finishing his Ph.D. and preparing to start a postdoctoral position at Northwestern University.

To my family.

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I have saved the best for last. I dedicate this thesis to my family for their love and support.

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CHAPTER 1

INTRODUCTION

New internet technologies over the past decade have been changing economies and societies around the world. These innovations, varying from information retrieval, social networks to electronic commerce, are creating a trend in information technology, namely, the merging of human collective behavior and technologies to create knowledge on a global scale.

Computer science, the main science behind this technology trend, is facing many great challenges. The traditional computing models of Turing machines, which assume that the designer has a full control on the input information and the execution of the program, are unrealistic in many modern applications. Primary examples of these applications include the internet routing networks that consist of multiple routers making independent decisions based on local information and several applications in electronic commerce where information is held by self-interested agents.

In the past decade an important line of research in theoretical computer science, known as *algorithmic game theory*, has emerged as an interdisciplinary research area between algorithms and game theory. This is a subfield of theoretical computer science that evolved from studying computer programs executed by stand-alone machines to complex systems involving a large number of agents, who pursue their own interests. Algorithmic game theory has become a natural research area that uses game theoretical approaches to investigate many problems in algorithm design and studies several concepts of game theory using algorithmic methods. Algorithmic game theory formulates new problems and develops novel solutions for relevant modern computer science applications.

This dissertation studies a general class of resource allocation games in computing systems, an important topic of algorithmic game theory. The focus of this study is on the design of decentralized mechanisms to allocate resources to self-interested agents. Game theoretical approaches for resource allocation problems overcome the drawbacks of centralized scheduling algorithms, where private information on the needs of agents is assumed to be available.

The main starting point of this dissertation is to study the *revenue* that can be obtained by the providers. This is an important question because in many cases the mechanisms are designed by providers who care about profit. Even when providers can only use a certain type of mechanisms, it is often the case that this class of mechanisms has a flexibility for providers to choose some parameters. In these scenarios provider will also adjust the parameters to maximize their revenue.

There is a large literature in resource allocation games and proportional sharing mechanism that we study in this dissertation. However, most of the works along this line focus on the *social welfare* of the system. Another difference of our studies, compared with the literature, is the solution concept that we use. In traditional mechanism design, revenue maximizing auctions are studied in Bayesian settings, where the type of each player is drawn from commonly known distributions. In this thesis we use the concept of *Nash equilibrium in full information settings*, where providers do not know users' private information or they need to use simple and natural mechanisms.

Mathematically, we study an abstract resource allocation game, where the resource constraints can be captured by a general polyhedron. This class of games captures a wide range of applications in computer science, varying from

sharing network bandwidth to scheduling jobs in data centers and distributing click-through resources in sponsored search. By taking a general approach, we can investigate different problems in a unified framework and use techniques developed in one problem for another. We now start with some simple examples to illustrate the content of the dissertation.

1.1 Illustrating Examples

In our resource allocation games, we assume that there are n users, and the goal of the game is to determine a real valued outcome $x_i \geq 0$ for each player i , which we think of as the player's level of activity or allocation. Each user has a non negative, monotone increasing and concave utility function $U_i(x_i)$. Each user maximizes his pay-off which is assumed to be the difference between the utility and his payment $U_i(x_i) - p_i$.

Fair Sharing for Single Resource and Equilibrium Price The simplest example in resource allocation games is the case of sharing an infinitely divisible resource of capacity 1 to a set of n users, each submits a non negative number (bid) b_i . The allocation x_i to user i is set *proportional* to b_i as follows:

$$x_i = \frac{b_i}{\sum_j b_j},$$

and the payment that user i will need to pay is b_i .

There are many other alternative mechanisms for this simple auction, such as the the first and second price auction, but the proportional sharing is a natural and simple mechanism. Most importantly, proportional sharing is scalable for

a wide class of users' utility functions, for example, any concave function $U_i(x_i)$, a scenario, where in the first or second price auctions users need to report the whole utility function.

The fair sharing mechanism above also provides a framework of recovering the *market cleaning price* via a decentralized implementation. The market cleaning price in this setting is an unit price p , such that when each user chooses to buy an x_i fraction of the resource to maximize his payoff $U_i(x_i) - px_i$, (equivalently, $U'_i(x_i) = p$ if $x_i > 0$) the total demand $\sum_i x_i$ is equal to the available resource, in our case $\sum_i x_i = 1$. Now, in proportional sharing, each user chooses a bid b_i and because of the proportional sharing rule, we always have $\sum_i x_i = 1$. Furthermore, when the mechanism reaches a Nash equilibrium, then the price per unit that each user pays is the same and equal to $\frac{b_i}{x_i} = \sum_i b_i$, which is not exactly the market cleaning price. However, it can be shown that when the number of users increases, this value approaches to the market cleaning price.

The proportional sharing mechanism also captures the effect of each individual user on the equilibrium price. Classical economic theory sometimes explains the market equilibrium by models consisting of infinitely many buyers where individual's strategy does not affect the price of the whole market. This assumption is not a reasonable in many settings. The proportional sharing mechanism overcomes this criticism by the fact that users' strategies do affect the unit price of resources. In other words, the users are "price anticipating".

More General Settings Another advantage of the fair sharing mechanism is that it can be extended to a much more general setting. Consider the following example. The service provider can either serve a single user (numbered 0) or a

set of other users (numbered from 1 to n). If randomization is allowed then this setting can be captured by the following inequality system

$$x_0 + x_i \leq 1 \quad \forall i \in [1, \dots, n].$$

This inequality system captures exactly the following network bandwidth sharing game. User 0 is interested in a path of bandwidth x_0 containing n different edges e_1, \dots, e_n , each with a capacity of 1. User i , $1 \leq i \leq n$, is only interested in a path containing single edge e_i . See Figure 1.1.

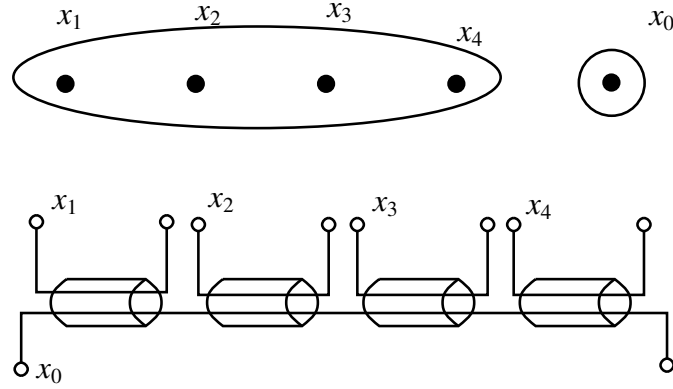


Figure 1.1: Network bandwidth sharing and downward close set system.

The proportional sharing mechanism can be extended for this setting as follows. User 0 bids a non negative number b_0^i on each edge of the graph and user i , $1 \leq i \leq n$, only bids b_i on the edge e_i . The mechanism will use the fair sharing on each link. User i , $1 \leq i \leq n$ pays b_i and gets

$$x_i = \frac{b_i}{b_i + b_0^i}.$$

User 0 pays $\sum_i b_0^i$ and gets

$$x_0 = \min_i \frac{b_0^i}{b_i + b_0^i}.$$

Similar to the case of a single resource discussed above, the sum of the bids on each link can be seen as the price of each resource, which is determined

by the demand of the users. It is also well understood that in the computer network setting, these prices have an interpretation of the average delay on each link [30, 29].

Proportional Sharing for a General Polyhedron In this thesis, we study this proportional sharing and extend it for an even more general setting that we call polyhedral environment. This will be defined more formally in Chapter 3. Intuitively, each edge in the example of the network above corresponds to a linear constraint of a general polyhedron. We will see later in Chapter 4 that, in the auction setting one can think of this generalized proportional sharing as a way to design competition among users. Moreover, this general problem captures many other applications, including the sponsored search auction. Our approach gives a rich model for this application because it can model complex externalities among advertisers.

1.2 The Questions and Contributions of the Thesis

Our first goal is to analyze the revenue of Nash equilibrium of the proportional sharing mechanism for general polyhedral environments. We consider a situation where the system consists of many users having similar demands and utilities. This is a natural scenario in many systems, such as the internet routing network and many internet auctions.

Question 1: With symmetry among competing users, what is the revenue and efficiency of the proportional sharing?

In Theorem 3.2, we show that both the efficiency and revenue converges to the optimal if the number of competing users increases.

The bounds on the efficiency and revenue in Theorem 3.2 are quite strong. But the case when there is no symmetry among users remains an important question. Although as we will see in Chapter 3, the proportional sharing always gives a near efficient allocation, the revenue can be very poor. An simple example is the case of bandwidth sharing game in Figure 1.1, where user 0 has a linear utility $U_0(x) = \epsilon \cdot x$ for a small ϵ and user i 's utility is $U_i(x) = x$. It is not hard to see that the proportional sharing described there only creates direct competition between users $i, 1 \leq i \leq n$ and user 0, thus user $i, 1 \leq i \leq n$ do not have incentive to pay high and therefore the revenue is low. For instance, if $\epsilon = 0$, then at Nash equilibrium, the revenue is 0. Thus, we come to the following question.

Question 2: How much revenue should we expect to get and how should we design a mechanism to get a competitive revenue when there is a lack of symmetry among competing users?

In Chapter 4, we answer this question by introducing a new revenue benchmark for the general auction setting (Definition 4.2) and show that one can combine the proportional sharing mechanism with a reserve price scheme to obtain a constant factor of this revenue benchmark (Theorem 4.4).

Questions 1 and 2 are concerned with the design of mechanisms that do not have any information on the valuation (utility) of the users. It has been recognized that in practice, providers try to learn the market demand and charge different prices for different market segments. This is commonly called price

discrimination [61]. Price discrimination is studied in many settings, including the full information games [59, 60] and asymmetric information games [5]. Price discrimination is usually used to increase the seller’s revenue, the effects of price discrimination on social welfare, however, are unclear.

We would like to understand this effect of discrimination *in proportional sharing* in a full information setting. To study this question, we introduce a generalization of proportional sharing mechanism, which we call the *weighted proportional sharing mechanism*. In this mechanism, each user i is allocated $x_i = \frac{b_i}{\sum_j b_j} C_i$, where the values C_i are decided by strategic providers to increase the revenue. Our third question is

Question 3: When the provider uses the weighted proportional sharing mechanism to discriminate among users, how much revenue can the provider get and what is the efficiency loss?

We show that the revenue of the weighted proportional allocation is nearly as good as the revenue under standard price-discrimination, where provider can charge different *unit prices* for different users. For linear user utility functions, the social welfare at Nash equilibria is at least $1/(1+2/\sqrt{3}) \approx 0.464$ fraction of the maximum social welfare, and this bound is tight (Theorem 5.4). We extend this result to a broader class of utility functions and to the case of many providers (Theorem 5.9).

In the application to sponsored search, our framework gives a different approach from the the General Second Price (GSP) auctions that is in common use by search engines. GSP is an algorithm for placing ads to ad-slots, where the bids of advertisers are multiplied by weights that can be different for different

advertisers and such weighted bids are used for placing the ads. The larger the weighted bid, the better the position that the ad gets. The reason to introduce these weights is explained by the term click-through rates r_{ij} , which is the probability that users click on ad i when it is placed at position j . It is commonly assumed that $r_{ij} = \alpha_i \times \beta_j$, where α_i is the quality of an ad i capturing how relevant the ad is to the search keyword, and β_j is the quality of the position: a large value of β_j is associated to a good position among the sponsored links.

The first disadvantage of this approach is that it is possible that in addition to the ad's quality, other ads that appear on the same page can affect its click-through rate (externalities). Thus, α_i cannot capture the real click-through rates. Second, the values α_i are given by search engines, it might be the case that these parameters are also designed strategically to maximize revenue.

The weighted proportional mechanism is studied for general polyhedral environment, a model that captures an application of sponsored search with complex externalities among ads. Also, in weighted proportional mechanisms, the weights are decided by profit maximizer providers as an analog to the structure of the General Second Price auctions.

1.3 Related Literature

Optimal Mechanism in Bayesian Settings Profit maximization in mechanism design has an extensive history beginning, primarily, with the seminal paper of Myerson [40] and similar results by Riley and Samuelson[50]. These papers study optimal mechanism design in Bayesian settings and the solution concept the Bayes-Nash equilibria. In this setting, players' types are assumed to

be drawn from commonly known distributions, and each player only knows about his own type. A Bayesian Nash equilibrium is a strategy profile that maps each player's type to an action such that, given this strategy each player maximizes their expected payoff over other players' distributions. In the optimal auction of Myerson, Riley and Samuelson, players are the bidders and the goal of the auctioneer is to design a mechanism to maximize the expected revenue. This material is by now standard and can be found in basic texts on auction theory [37, 25].

Prior-free Truthful Mechanism Design The optimal mechanism in Bayesian settings highly depends on the distributions of bidders' type. In many applications, these distributions are hard or impossible to obtain. Prior-free auctions have recently been of much research focus because of the need for more robust auctions that do not depend on the underlying distributions of bidders' valuations. The main constraint in this line of work is to require the mechanism to be "truthful", that is, it is best for bidders report their true type regardless what other bidders do. This approach is considered in economics literature as "detail-free" or "robust" mechanism design [8]. In computer science the approach was first considered by [12] and followed by a large literature [14, 16, 15]. The truthful condition is strong, furthermore, this framework does not provide a nice characterization for the optimal revenue as in the Bayesian setting. The works in [12, 14, 16, 15] define revenue benchmarks and design mechanisms that approximate these benchmarks.

Nash Implementation in Full Information Settings This thesis takes a different approach from the two lines of research described above. We use the theory

of Nash implementation in full information settings. In this setting, players have the complete information about each other. This does *not* mean that the designer knows this information. In Chapter 3, 4, we assume that the designer does not have any information about the users' utility, he only knows the set of possible outcomes. The literature on Nash implementation of full information games is large, initiated with the seminal work Maskin [33], for which he won the Nobel prize in 2007. For more on related works in the area, see the surveys [33, 34, 48, 32]. This literature, however, is mostly concerned about implementation for the goal of maximizing social welfare. This is where this dissertation differs from previous work. We focus on the revenue can be obtained in Nash equilibria.

Proportional Sharing Mechanism The classical proportional sharing mechanism is introduced and studied by Kelly [24]. There is a large literature studying various aspects of the proportional sharing mechanism, including robustness, convergence of response dynamics, efficiency and practicability [62, 13, 55, 22, 20, 18, 30, 49]. Johari and Tsitsiklis [20] show that, when the utilities U_i are concave, then at Nash equilibria the social welfare is at least $3/4$ times the social welfare of the most efficient allocation. The revenue of proportional sharing is studied by the author with Éva Tardos and Milan Vojnović in [42, 43, 44, 41].

Sponsored Search Auctions Sponsored search is a form of advertising, typically sold at auctions where merchants bid for positioning alongside web search results. This is one of the fastest growing, most effective and profitable forms of advertising, that has attracted researchers in both computer science and economics [28, 10, 6, 58, 3]. Our work connects the basic proportional sharing

mechanisms to the applications of sponsored search. Our framework captures complex externalities, an important feature of sponsored search auctions.

Mechanism Design with Many Sellers One of our results in this thesis is for the case of multiple providers. This is an exciting direction in mechanism design. Mechanism design for multiple providers is complex and not very well understood. Many standard techniques such as revelation principle fails in this environment. For more details on recent development of this area see the survey of D. Martinmort [31] and recent works of M. Pai [45, 46].

Structure of the Thesis

The dissertation has 6 chapters. In Chapter 2 we give some basic notations and concepts that will be used throughout the thesis, we also describe applications of the general polyhedral environments. Chapter 3, 4 and 5 give answers to the three questions discussed at the beginning of this chapter. Chapter 6 concludes the dissertation with future research directions.

CHAPTER 2

PRELIMINARIES: BASIC NOTATIONS AND SETTINGS

2.1 Basic Notations

Providers and Users The general resource allocation games we study consist of multiple *providers* and *users*. Providers own the resources and use some types of mechanisms to allocate the resources to the users. In this thesis, depending on the context, we sometimes use *sellers*, *auctioneers* for providers or *buyers*, *bidders* for users. In Chapter 3 and Chapter 4, we investigate the case of a single provider. The general case of many providers is considered in Chapter 5. We denote by n the number of users in our system.

Allocation Vectors The resource that a user i gets is expressed as a non negative real value x_i indicating the user's level of activity or allocation. We call \vec{x} an *allocation vector*. Usually, providers have limited resources, and the allocation vectors need to satisfy some constraints. In this thesis, we assume that the provider knows the set of all possible allocation vectors \vec{x} .

Users' Utilities Each user has an *utility function* $U_i(x_i)$ on the amount of resource x_i that he gets. We will assume that all U_i are *non negative, monotone increasing and concave*, and $U_i(0) = 0$. The concavity condition is a traditional assumption in the literature to capture the diminishing returns property of utilities. This is one of the most common assumptions used in economics literature.

Single Parameter Setting We sometimes focus on a special case of utility functions, namely, linear utilities: $U_i(x_i) = v_i x_i$. In this situation, we call the setting as *single parameter*, because each utility can be represented as a single number $v_i \geq 0$. We call v_i the (private) *valuation* of user i . In designing a mechanism, we assume that the provider does not know the valuations of users.

Mechanisms In a mechanism, each user i has a *message space* M_i to report or signal to the provider and other users about his type. Based on the reported messages (m_1, \dots, m_n) , $m_i \in M_i$ from all the users, the provider allocates the resource x_i and asks for a *payment* p_i from the user i . Thus, \vec{x}, \vec{p} are functions on the domain $\prod_{i=1}^n M_i$. We assume that there is also an option for each user not to participate in the mechanism. This can be encoded as a special \emptyset message in each M_i .

Quasi-linear Payoff In this thesis, we assume users have quasi-linear payoff, which is the difference between the utility and the payment: $U_i(x) - p_i$.

Nash Equilibrium Nash equilibrium is the solution concept mostly considered in this thesis. We assume a vector \vec{m} to be a Nash if assuming no other user want to change their message, it is best for user i to keep his m_i to maximize the payoff, which is $U_i(x_i) - p_i$. Because there is a “not participate” option for each user, at Nash equilibrium, for every user i , we have $U_i(x_i) \geq p_i$.

Dominant Strategy Equilibrium A Nash equilibrium is called dominant strategy if it is best for each user i to keep his m_i no matter how other users report their messages. Dominant strategy is a stronger solution concept than

that of Nash equilibrium. It has been showed that every mechanism with dominant strategy equilibrium can be implemented by a mechanism, where each user report directly their utility, or in the single parameter setting to report their valuation. This mechanism is called *truthful mechanism*.

Revenue The revenue of a mechanism is the total payment of all users $\sum_i p_i$. Depending on the solution concepts, one can talk about the revenue of a Nash equilibrium or of a truthful mechanism.

Social Welfare, Price of Anarchy The social welfare is defined as the total of users' utility: $\sum_i U_i(x_i)$. In quasi linear-payoff model, the social welfare is the sum of users' payoffs and the total revenue obtained by the providers. In many cases, we would like to compare the social welfare at Nash equilibrium with the optimal social welfare. The ratio between the worst social welfare of a Nash and the optimal social welfare is call the *price of anarchy*.

2.2 Polyhedral Environments and Applications

In the following we will describe a general environment that we call polyhedral environment. This is a general type of constraints on the resources, that capture a wide range of applications in computer science.

The provider has *polyhedral* constraints on the resource. That is, the allocation vector \vec{x} that the provider can allocate needs to be in a convex set of a form $\{\vec{x} \in \mathbb{R}_+^n : A\vec{x} \leq \vec{1}\}$, where A is a non negative matrix.

Note that any polyhedron of the form $\{A'\vec{x} \leq \vec{c}, x \geq 0\}$, where A' is a non negative matrix and \vec{c} is a non negative vector, can be normalized to the form of $\{A\vec{x} \leq \vec{1}, x \geq 0\}$.

Network Bandwidth Sharing The most natural example is the bandwidth sharing game, where each provider owns a network of capacitated links, each user i is sending traffic along a path P_i and x_i is the data transfer rate for user i . In this case we have a resource constraint associated to each link e : $\sum_{i: e \in P_i} x_i \leq c_e$ where c_e is the capacity of link e .

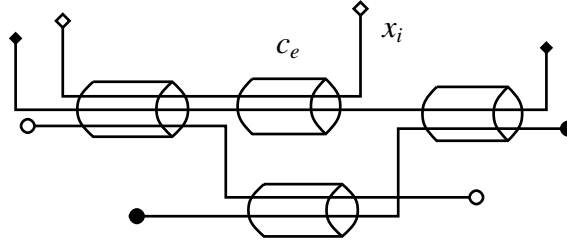


Figure 2.1: Network bandwidth sharing.

Keyword Auctions The general convex constraints can also capture a general model of keyword auctions. This is the main application to be considered in Chapter 5. The auction is for a single keyword, and there are n advertisers bidding to have their ad appear as a sponsored search result. There are finite set of outcomes, depending on which bidder gets displayed in which position. We describe each of these outcomes as a n dimensional vector whose coordinates are the expected number of clicks that the corresponding advertiser gets. More precisely, let $\vec{x}^1, \dots, \vec{x}^N$ be all the possible outcome vectors, and $\vec{x}^k = (x_1^k, \dots, x_n^k)$, where x_i^k is the expected number of clicks that advertiser i receives at outcome k . To think of keyword auction as a convex resource allo-

cation, we need to allow randomization in the allocation of bidders to positions. Choosing between the deterministic allocations by the probability distribution $\vec{p} = (p_1, \dots, p_N)$, we have that $\sum_j p_j \vec{x}^j$ is the vector whose coordinates correspond to the expected number of clicks of an advertiser. Now the set of expected allocation vectors obtained this way is exactly the convex hull $\text{conv}(\vec{x}^1, \dots, \vec{x}^N) = \{\vec{x} : \vec{x} = \sum_j p_j \vec{x}^j, p_j \in [0, 1] \text{ for every } j \text{ and } \sum_j p_j = 1\}$. This way of modeling keyword auctions will be discussed in more details in Chapter 5. We will show that the convex hull of \vec{x}^k can be seen as a special case of our polyhedral environment under a natural assumption.

Single Parameter Auction Represented by a Downward-closed Set System

This application will be discussed in more detail in Section 4. In this setting each agent has a private valuation for receiving service and there is a set system representing feasible sets. A feasible set is a set of agents that can be served simultaneously. For example in auction for single item the feasible set system

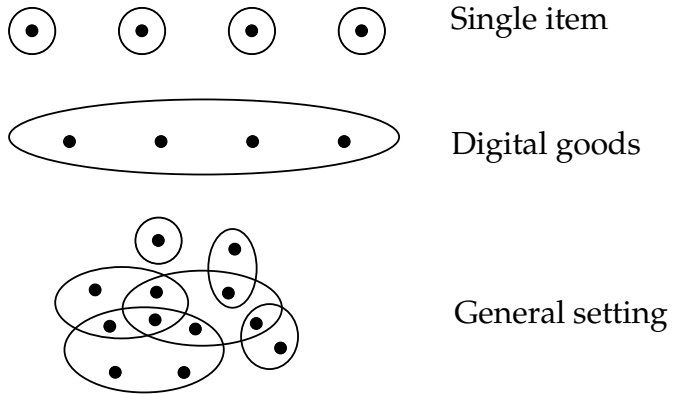


Figure 2.2: General auction setting.

contains singleton. We focus on the typical case of downward-closed environment where every subset of a feasible set is again feasible. Another example of such an environment is a combinatorial auction with single-minded bidders,

where feasible sets correspond to subsets of bidders seeking disjoint bundles of goods. A more general example is a combinatorial auction with single-value bidders, each of them has an utility of a single value, v_i , when he obtains one of many possible sets. It will be shown latter that the randomized outcomes of this environment can be captured by our general polyhedral setting.

Scheduling Jobs in Data Centers This is a problem of allocating data center resources to users. In this application, typically each user needs to finish a job which requires reading many different blocks of data across machines in a data center. Let D_i^j be the amount of data of type j that job i needs to process and s_i^j be the speed that job i can process data of type j . Thus, the time to read this data is D_i^j/s_i^j . The finishing time of job i is t_i , which is the maximum processing time of the job across all types of data that it requests. One can consider the model when each job i tries to maximize the utility $U_i(1/t_i)$. Typically, data centers are complex systems consisting of many clusters of machines and data has many copies across the clusters. The constraints on s_i^j are complex, but in many cases it can be captured by convex constraints. Therefore, the allocation vector \vec{x} can also be captured by convex constraints. In this example, it is unrealistic to design a mechanism that requires every job to know exactly the complex constraints on \vec{x} . Simple mechanisms are crucial in these applications.

CHAPTER 3

PROPORTIONAL SHARING FOR POLYHEDRAL ENVIRONMENTS

The fair sharing mechanism was motivated by the need for a simple and easy to implement mechanism for the resource sharing problem on the Internet. This mechanism is now quite well studied and has been used to implement many internet routing protocols. The design of internet congestion control protocols is based on several ideas varying from using auctions to simple pricing. But these proposals share the basic goal of maximizing social welfare. The idea is to implement a simple lightweight mechanism that helps arrange the socially optimal sharing of resources.

Congestion pricing [23, 53], has emerged as a natural way to decide how to share bandwidth in a congested Internet. While maximizing social welfare is important to keep customers subscribed to the system, we believe that revenue should also be considered. Once a mechanism gets implemented, the network managers will try to take advantage of the users, and aim to maximize income, and will no longer only think of the mechanism as a way to arrange the best use of the network by maximizing social welfare. As a result, it is important that we also understand the revenue generating properties of the proposed mechanisms.

In this chapter we investigate this question in the context of a proportional sharing mechanism of Johari and Tsitsiklis [20] that generalizes the fair sharing for *general polyhedral environments*. Our main motivation is to study the performance of this mechanism in setting where there is a high symmetry among competing users. This is a natural assumption, especially in the networking scenario where users are often classified into few categories : “small” or “heavy”

users, “uploaders” or “downloaders”. Our main question is:

Under a symmetry assumption how the fair sharing mechanism achieve both goals of revenue and efficiency?

Results We show that with few assumptions, which we will explain more formally in Section 3.2, we can obtain good bounds on both efficiency and social welfare. We develop a new technique for analyzing such allocation games, and bound the revenue. Our technique for bounding the revenue uses the similarity between the condition of Nash equilibriums of the game and the dual of a certain linear program. We show that the game approximately maximizes the revenue of the auctioneer, with the approximation ratio tending to 1 if players’ utilities are linear and the number of identical players increases. In a more general class of utilities satisfying $U(2x) > \alpha U(x)$ for some constant $\alpha > 1$, the approximation ratio of the revenue will tend to $\alpha - 1$. We also strengthen the efficiency result to show if there are k users of every type than the efficiency is at least $(1 - \frac{1}{4k})$ times the social welfare of the most efficient allocation, i.e., the efficiency tends to the optimal as the number of identical players increases. Our main theorem can be claimed more precisely as follows:

MAIN THEOREM *Given a constant $\alpha > 1$, and an integer $k \geq 2$, under the assumption that each player’s utility satisfies $U(2x) > \alpha U(x)$ and for each player type, there are at least k players (defined formally in section 3.2), the fair sharing mechanism (defined in section 3.1) obtains both approximately maximum efficiency, and approximately maximum revenue. The efficiency is at least $(1 - \frac{1}{4k})$ times the optimal efficiency and the revenue is at least $(\alpha - 1)(1 - \frac{1}{k})$ the optimal revenue.*

Remark Note that this bound is very strong when utility is linear (and so $\alpha = 2$). For this case we have the revenue of the mechanism is at least $f(k) = (1 - \frac{1}{k})$ times the optimal. Already when there are 2 players of each type (when $k = 2$) the mechanism achieves $\frac{7}{8}$ times the optimal efficiency and half of the maximum revenue.

Organization of the Chapter In Section 3.1 we describe the mechanism in the polyhedral environment. Section 3.2 discusses the bound on the revenue and the efficiency of this game. Section 3.3 discusses the related literature.

3.1 Proportional Sharing in General Polyhedral Environments

In this section we describe the fair sharing mechanism for the general class of games introduced in Chapter 2. The mechanism is an extension of the mechanisms introduced by Kelly [23], Johari and Tsitsiklis [20]. Let E denote the set of constraints (the rows of A). For simplicity of notation, we assume that $u_e = 1$ for each $e \in E$ by normalizing each row. We will use α^e to denote the row e of matrix A , which we will also call constraint e . We now have the following description of the set of feasible allocations:

$$\begin{aligned} \sum_i \alpha_i^e x_i &\leq 1 \text{ for all } e \in E, \\ x_i &\geq 0. \end{aligned} \tag{3.1}$$

The Mechanism When sharing a single resource with constraint $\sum_i x_i \leq 1$ the fair sharing [23] mechanism requires that each player j submits a bid b_j , the amount of money she wants to pay, and the resource is allocated proportional

to the bids, as $x_j = b_j / \sum_i b_i$. We can think of $\sum_i b_i$ as the unit price p of the good. The allocation is derived from this unit price, as user j gets $x_j = b_j / p$ amount for the cost $w_j = b_j$ at this price.

To extend this mechanism to a single constraint with coefficients $\sum_i \alpha_i x_i \leq 1$, we again require that each player j submit a bid b_j , her willingness to pay, and view $p = \sum_i b_i$ as the unit price of the good. Recall that α_j^e is the rate at which user j uses resource e , so user j needs $\alpha_j x_j$ allocation for a value x_j . At the unit price of p she gets $\alpha_j x_j = b_j / p$ allocation, and hence we need to set $x_j = b_j / (\alpha_j p) = \frac{b_j}{\alpha_j \sum_i b_i}$, and she will have to pay $w_j = b_j = \alpha_j x_j p$.

For environments with more constraints, Johari and Tsitsiklis [20] extends the fair sharing mechanism by requiring that users submit bids b_j^e separately on each resource e . As before, we can view the sum of bids $p^e = \sum_i b_i^e$ as the unit price of resource e , and allocate the resource at this price. This allocation limits the value x_j for user j to at most $x_j^e = b_j^e / (\alpha_j^e p^e)$. The idea is to ask users to submit bids b_j^e for each resource e , allocate the resources separately, make user j pay $w_j = \sum_e b_j^e$, and then set $x_j = \min_{\{e: \alpha_j^e > 0\}} x_j^e$.

We need to extend this mechanism to be able to deal with resources that are under-utilized. Some constraints e may not be binding at any solution, and the fair sharing method does not deal well with such constraints: users will want to bid arbitrary small amounts as there is too much of the resource. To deal with such constraints, we allow each player to request an amount r_j^e without any monetary bid. For each resource e if the price is 0 (that is $p^e = \sum_i b_i^e = 0$) and $\sum_i \alpha_i^e r_i^e \leq 1$ (the requested rates can all be satisfied) then we setting $x_j^e = r_j^e$ for all j .

The mechanism can be described formally as follows:

DEFINITION 3.1 (Generalized Proportional Sharing) *Each player j submit a bid b_j^e and a request r_j^e for each resource e . For resource e we use the following allocation:*

- *If $\sum_i b_i^e > 0$ then $x_j^e = \frac{b_j^e}{\alpha_j^e(\sum_i b_i^e)}$ for $\forall j$*
- *If $\sum_i b_i^e = 0$ and $\sum_i \alpha_i^e r_i^e \leq 1$ then $x_j^e = r_j^e$ for $\forall j$*
- *Else, set $x_j^e = 0$ for $\forall j$.*

For each player j , the amount of money that she needs to pay is $w_j = \sum_e b_j^e$ and the final allocated $x_j = \min_{\{e: \alpha_j^e \neq 0\}} x_j^e$.

Price Taking Strategy Kelly [23] has considered a version of this “game” when prices are assigned by the network, and users are “price takers” in the sense that they act to optimize their value at the given prices. We can also view our fair-sharing game as a pricing game, but in our game the prices are determined as part of the game. However, it is useful to compare the mechanism above with a game where players behave as price takers.

Consider an equilibrium of the game, it must be the case that $x_j^e = x_j$ for all resources e that costs money, or otherwise player j can reduce her bid b_j^e without affecting her allocation. One way to think about the mechanism above is the following: Players decide on each resource (constraint) a price $p^e = \sum_j b_j^e$; now players have to pay for each resource e at its unit price p^e . To make sure a player i gets enough of resource e to have a share of x_i , he needs $\alpha_i^e x_i$ of the

resource, and hence needs to pay $p^e \alpha_i^e x_i$ for resource e . In order to get all the needed resources player i must pay a unit price of $\sum_e \alpha_i^e p^e$ for his resource.

Now, if we assume that the price p^e are given, then for each player i the unit price is fixed. Therefore to maximize her utility, player i will maximize his utility, that is $U_i(x_i) - \sum_e p^e \alpha_i^e x_i$. Taking the derivative in x_i to determine the optimal value for user i we see that user i will choose to buy an x_i such that: the derivative $U'_i(x_i)$ is equal to the unit price or in the case $U'_i(0)$ is less than the unit price, she will choose not to buy any resource. We rewrite this as follow:

$$U'_i(x_i) = \sum_e \alpha_i^e p^e \quad \text{OR} \quad x_i = 0 \quad \text{if} \quad U'_i(0) < \sum_e \alpha_i^e p^e. \quad (3.2)$$

Condition for Nash Equilibriums In our mechanism, the prices p^e are not fixed. They are the sum of all the bids on each constraint, which are given by strategic players. As a result, the Nash condition given below is slightly different from (3.2). In the allocation game, the price is a function of the bids, and this induces the players to “shade” their bid for the resource, getting a bit less resource at a smaller price. Johari and Tsitsiklis [20] prove that a Nash equilibrium exists and give the following conditions. In this condition, observe that the differences between the Nash condition (3.3) and the condition (3.2) are the terms $\frac{1}{(1-\alpha_j^e x_j)}$. We will show later that using the competitiveness condition (defined in Section 3.2), these terms are small.

THEOREM 3.1 ([20]) *If the utility function of each player is increasing, differentiable and concave, then there always exists a Nash equilibrium.*

An allocation x a bid and a request vector b, r is a Nash solution if and only if:

$$\begin{aligned} \sum_i \alpha_i^e x_i &\leq 1; \quad x_i \geq 0 \text{ for all } e \in E, \\ U'_j(x_j) &= \sum_e \frac{p^e \alpha_j^e}{(1 - \alpha_j^e x_j)}, \text{ if } x_j > 0 \text{ or} \\ x_j &= 0 \text{ if } U'_j(0) \leq \sum_e p^e \alpha_j^e; \text{ where } p^e = \sum_i b_i^e. \end{aligned} \quad (3.3)$$

Proof. To simplify the presentation, and without loss of generality, we will assume that each resource e has at least two dedicated users who only needs resource e , and who have small, but linear utility ϵx . These users will guarantee that no resource is under-utilized, but will not change either the optimal allocation of the Nash equilibrium substantially. Using this assumption, we can never have $\sum_i b_i^e = 0$ for any resource e . To get the result we need to take the limit as the rate ϵ of the utility of the extra users tends to 0 (see [20]).

Next we analyze the condition for an equilibrium for this game. We will use these conditions to show that an equilibrium always exists. Consider a set of bids b_i^e , and a resulting allocation x , where player i gets allocation x_i . When is this allocation at equilibrium? For each resource e we use $p^e = \sum_i b_i^e$, the sum of the bids, as the unit price of the resource (recall that we normalized constraints, so there is 1 unit of every resource available).

Now consider the optimization problem of a player j assuming bids b_i^e for all other players are set. The player j is interested in maximizing her utility at $U_j(x_j) - \sum_e b_j^e x_j$. At equilibrium, it must be the case that $x_j^e = x_j$ for all resources e that costs money, or otherwise player j can reduce her bid b_j^e without affecting her allocation. So we can think of the player's optimization problem as dependent on one variable x_j , the allocation she will receive. What bid does player j have to submit for a resource e to get allocation $x_j^e = x_j$? Bids must satisfy the

following condition:

$$\text{If } b_j^e > 0 \text{ then: } \alpha_j^e x_j = \frac{b_j^e}{\sum_i b_i^e}.$$

Assuming all other bids b_i^e are fixed, we can express the bid b_j^e needed as follows.

$$b_j^e(x_j) = \frac{\alpha_j^e x_j \sum_{i \neq j} b_i^e}{1 - \alpha_j^e x_j}.$$

Note that this expression assumes that $\alpha_j x_j < 1$, that is, j is not the only user of the resource at equilibrium. It is not hard to see that this is guaranteed by having at least two dedicated users for each resource.

User j will want to choose x_j to maximize her utility. For this end, it will be useful to express the derivative of the bid b_j^e when viewed as a function of x_j . We get the following (again assuming $\alpha_j x_j < 1$):

$$\frac{\partial}{\partial x_j} b_j^e(x_j) = \frac{\alpha_j^e \sum_{i \neq j} b_i^e}{(1 - \alpha_j^e x_j)^2}.$$

Substituting $\sum_{i \neq j} b_i^e = p^e(1 - \alpha_j^e x_j)$ and simplifying we get that

$$\frac{\partial}{\partial x_j} b_j^e(x_j) = \frac{p^e \alpha_j^e}{1 - \alpha_j^e x_j}.$$

Now consider the optimization problem of player j . She wants to maximize her utility $U_j(x_j) - \sum_e b_j^e$, which can now be expressed as

$$U_j(x_j) - \sum_e \frac{\alpha_j^e x_j \sum_{i \neq j} b_i^e}{1 - \alpha_j^e x_j},$$

as a function of the single variable x_j . Note that this is a concave function of x_j . The maximum occurs at a value x_j , where the derivative of this function is 0, or if the derivative is negative everywhere, maximum occurs at $x_j = 0$. Using the derivatives we computed above, we get the derivative of user j 'th utility as a function of her allocation x_j to be

$$U'_j(x_j) - \sum_e \frac{p^e \alpha_j^e}{(1 - \alpha_j^e x_j)}.$$

This derivative is a strictly decreasing function, so we have the following Nash condition:

$$\begin{aligned} \sum_i \alpha^e x_i &\leq 1; \quad x_i \geq 0 \text{ for all } e \in E \\ U'_j(x_j) &= \sum_e \frac{p^e \alpha_j^e}{(1 - \alpha_j^e x_j)}, \text{ if } x_j > 0 \text{ or} \\ x_j = 0 &\text{ if } U'_j(0) \leq \sum_e p^e \alpha_j^e; \text{ where } p^e = \sum_i b_i^e. \end{aligned}$$

To see that there is always a Nash equilibrium, observe the game we define above is a concave n-person game: each payoff function is continuous in the composite strategy vector \vec{b}_i , and the strategy space of each user is a compact, convex, nonempty subset of $\mathbb{R}^{|E|}$. Applying Rosen's existence theorem [51] (proved using Kakutani's fixed point theorem), we conclude that a Nash equilibrium w' exists for this game.

By this, we finished the proof. ■

3.2 Revenue and Efficiency of Proportional Sharing

In this section we analyze the revenue and efficiency of a Nash equilibrium. In the rest of the section we will use the variable x as a solution of the Nash condition (3.3). To evaluate the outcomes of the game x , we will compare the social welfare and the revenue with the optimal social welfare, which can be written as an optimum of the following a linear program. In this program to avoid using x as a solution of (3.3), we use new variable z_i for the amount of resource that buyer i gets.

$$\max \quad OPT = \sum_{i=1}^n U_i(z_i)$$

$$\begin{aligned} \text{subject to} \quad & \sum_i \alpha_i^e z_i \leq 1; \quad \forall e \in E \\ & z_i \geq 0. \end{aligned} \tag{3.4}$$

We denote z^* as a solution of the program above. We have $OPT = \sum_i U_i(z_i^*)$.

As already mentioned in the introduction, we need to make two assumptions to be able to get a reasonable bound on the revenue. First we assume that the players' utility functions grow at a reasonably steady rate. Second, we assume that there are at least $k \geq 2$ players of each type.

DEFINITION 3.2 (ASSUMPTION(α, k)) *The two assumptions are*

- **Growing Utilities:** *The utility function $U_j(x)$ of all users j is non negative, increasing, differentiable, concave, further, U_i satisfies: $U_i(2x) \geq \alpha U_i(x)$ for some $\alpha > 1$.*
- **Competitiveness:** *We say that the type of a player j is her utility function $U_j(x)$ and the rate at which she needs the resources, the coefficients α_j^e for all resources e . We assume that there are at least k players of every type: that is for every player j , there are at least $(k - 1)$ other players with the same type.*

In the context of bandwidth sharing, the second assumption means that for each player j , there are at least $k - 1$ other players with the same utility function and the same path.

The main result of the chapter is the following:

THEOREM 3.2 *Under Assumption(α, k), the mechanism defined in section 3.1 approximately maximizes both efficiency and revenue. The loss of efficiency is bounded by a fraction of $\frac{1}{4k}$ and the revenue is at least $(\alpha - 1)(1 - \frac{1}{k})$ times the optimal revenue.*

We prove this theorem in the rest of the section (by Theorem 3.7 and Theorem 3.11). We first introduce an Linear Program Duality technique and give some intuitions about this approach in the next subsection.

3.2.1 The Primal Dual Approach

The condition for Nash equilibriums can be intuitively understood as if there were a common price p^e on each constraints and for each buyer i the unit price that he needs to pay is the weighted sum of these prices with the coefficients α_i^e . At Nash equilibrium, $x_i = 0$ if $U'_i(0)$ is less than $\sum_e \alpha_i^e p^e$, otherwise $U'_i(x_i)$ can be approximated by this weighted sum. If we consider the prices p^e as variables then this condition on p^e is similar to the complementary slackness condition of a certain linear program. Using this observation, we consider the following linear program and its dual, where x is a given an Nash equilibrium.

$$\begin{aligned} \text{PRIMAL} \quad & \max \sum_i U'_i(x_i) z_i \\ & \text{subject to: } \sum \alpha_i^e z_i \leq 1; \quad z_i \geq 0. \end{aligned} \tag{3.5}$$

$$\begin{aligned} \text{DUAL} \quad & \min \sum_i y^e \\ & \text{subject to: } U'_i(x_i) \leq \sum \alpha_i^e y^e; \quad y^e \geq 0. \end{aligned} \tag{3.6}$$

LEMMA 3.3 (Weak Duality) *Given a z_i and y^e feasible solutions for the Primal and the Dual program respectively, we have: $\sum_e y^e \geq \sum_i U'_i(x_i) z_i$. ■*

To prove the bound on the revenue, we observe that the price vector p^e of a Nash satisfying the condition (3.3) almost satisfies the condition of the DUAL

program. The extra terms $\frac{1}{1-\alpha_i^e x_i}$ in the Nash condition can be bounded by a constant. Furthermore, the convex program of maximizing social welfare is similar to the PRIMAL program. The difference between these programs is the objective function. And it will be shown later that using the growing property of the utility functions, these two objective functions are close to each other. Thus, with the duality lemma, we can get a connection between a the revenue of a Nash equilibrium and the optimal social welfare.

To prove the bound on the efficiency, based on the Nash condition, we will introduce new game on each constraint. The bound on the efficiency of each of these separated games is much easier to check.

3.2.2 Bound on the Revenue

We now prove that the revenue at a Nash equilibrium is at least $(\alpha - 1)(1 - \frac{1}{k})$ of the optimal. As mentioned before we use the optimal social welfare as an upper bound on the revenue. Recall that we use z^* as an allocation maximizing the social welfare. To compare the revenue $\sum_e p^e$ with $\sum_i U_i(z_i^*)$, we first show that $\frac{p^e}{1 - \frac{1}{k}}$ is feasible for the Dual program (3.6) and therefore:

$$\sum_e \frac{p^e}{1 - \frac{1}{k}} \geq \sum_i U'_i(x_i) z_i^*,$$

because z^* is clearly feasible for the Primal program (3.6). Next, using the growing property of the utility functions, we prove that

$$\sum_i U'_i(x_i) z_i^* \geq (\alpha - 1) \sum_i U_i(z_i^*).$$

Combining these two inequalities, we obtain:

$$\sum_e \frac{p^e}{1 - \frac{1}{k}} \geq (\alpha - 1) \sum_i U(z_i^*) \Rightarrow \sum_e p^e \geq (\alpha - 1)(1 - \frac{1}{k}) OPT,$$

which is what we need to prove.

In order to show that $\frac{p^e}{1-\frac{1}{k}}$ is feasible, we first observe that the equilibrium is not known to be unique in the general, however, players of identical type must get identical allocation:

LEMMA 3.4 *If two players i and j have the same type, then in any Nash equilibrium, they get the same allocation.*

Proof. By the Nash equilibrium conditions (3.3) both x_i and x_j are 0 if $U'_i(0) = U'_j(0) < \sum_i \alpha_i^e p^e$ and otherwise both are the unique solutions equation of Nash in (3.3). (The function on the left hand side of (3.3): $U'_j(x_j)$ is a decreasing function, meanwhile the function on the right hand side $\sum_e \frac{p^e \alpha_j^e}{1-\alpha_j^e x_j}$ is an increasing function.) ■

We now can prove the following lemma:

LEMMA 3.5 $\frac{p^e}{1-\frac{1}{k}}$ is feasible for the Dual program.

Proof. Since for every buyer i there are other $k-1$ buyers of the same type, and due to Lemma 3.4, these buyers get identical allocation. Therefore, for each constraint e , we have $1 \geq \sum_j \alpha_j^e x_j \geq k \alpha_i^e x_i$. Thus $\alpha_i^e x_i \leq \frac{1}{k}$, and hence $\frac{1}{1-\alpha_i^e x_i} \geq \frac{1}{1-\frac{1}{k}}$.

From this and the Nash condition we have: $U'_i(x_i) \geq \sum_e \alpha_i^e p^e \frac{1}{1-\alpha_i^e x_i} \geq \sum_e \alpha_i^e p^e \frac{1}{1-\frac{1}{k}}$. This shows that $\frac{p^e}{1-\frac{1}{k}}$ is feasible for the Dual program (3.6). ■

We now prove the second inequality needed for bounding the revenue:

LEMMA 3.6 $\sum_i U'_i(x_i) z_i^* \geq (\alpha-1) \sum_i U_i(z_i^*) = (\alpha-1)OPT$.

Proof. The objective function of the primal linear program and the social maximizing program are $\sum_i U'_i(x_i)z_i$ and $\sum_i U_i(z_i)$, respectively. To compare these two functions at z^* , we use the tangent line $V_i(z_i)$ of the utility function at x defined as $V_i(z_i) = U'_i(x_i)z_i + (U_i(x_i) - U'_i(x_i)x_i)$. This is a line going through $(x_i, U_i(x_i))$ and is above $U_i(z_i)$ as U_i is a concave function. See Figure 3.1.

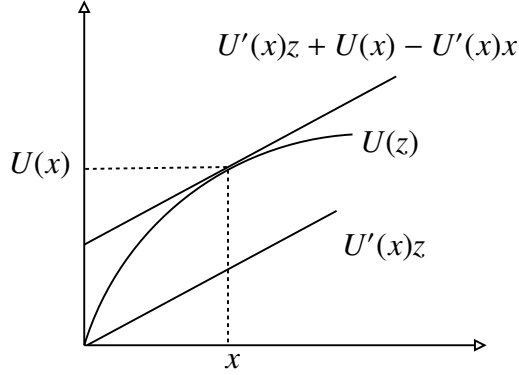


Figure 3.1: The shape of the utility functions.

Observe that for each i the function $f(z_i) = U'_i(x_i)z_i$ could be smaller than the function $U_i(z_i)$, but using the function V_i at z_i^* we get:

$$U'_i(x_i)z_i^* + U_i(x_i) - U'_i(x_i)x_i \geq U_i(z_i^*) \Rightarrow U'_i(x_i)z_i^* \geq U_i(z_i^*) - (U_i(x_i) - U'_i(x_i)x_i).$$

And summing this over all i :

$$\sum_i U'_i(x_i)z_i^* \geq \sum_i U_i(z_i^*) - \sum_i (U_i(x_i) - U'_i(x_i)x_i) = OPT - \sum_i (U_i(x_i) - U'_i(x_i)x_i). \quad (3.7)$$

Now, we need to get a bound on $\sum_i (U_i(x_i) - U'_i(x_i)x_i)$. To do that we will need the growing property of the utility functions. Because U_i is concave function $U'_i(x_i)$ is decreasing. We have:

$$U_i(2x_i) - U_i(x_i) = \int_{x_i}^{2x_i} U'_i(t)dt \leq \int_{x_i}^{2x_i} U'_i(x_i)dt = U'_i(x_i)x_i.$$

Using the assumption $U_i(2x) \geq \alpha U_i(x)$, we obtain:

$$U'_i(x_i)x_i \geq U_i(2x_i) - U_i(x_i) \geq (\alpha - 1)U_i(x_i).$$

Now because U_i are concave and $U_i(0) = 0 \quad \forall i$, we have $U_i(x_i) \geq U'_i(x_i)x_i$. Thus we obtain from the previous inequality:

$$0 \leq U_i(x_i) - U'_i(x_i)x_i \leq (2 - \alpha)U_i(x_i).$$

Summing over i , we get a bound on the difference between the objective function of the program (3.4) and the program (3.5) at x as follow:

$$0 \leq \sum_i (U_i(x_i) - U'_i(x_i)x_i) \leq (2 - \alpha) \sum_i U_i(x_i).$$

However, z^* is an optimal solution of (3.4), therefore $0 \leq \sum_i U_i(x_i) \leq \sum_i U_i(z_i^*)$.

Thus combining with the previous inequality we have:

$$0 \leq \sum_i (U_i(x_i) - U'_i(x_i)x_i) \leq (2 - \alpha) \sum_i U_i(z_i^*) = (2 - \alpha)OPT. \quad (3.8)$$

Combining (3.7) and (3.8) we have:

$$\sum_i U'_i(x_i)z_i^* \geq OPT - (2 - \alpha)OPT = (\alpha - 1)OPT.$$

This is indeed what we need to prove. ■

Thus, as discussed before, combining Lemma 3.5 and Lemma 3.6 with the Duality lemma, we have proved the following theorem:

THEOREM 3.7 *Under the assumption (k, α) , the revenue that a Nash equilibrium achieves is at least $(\alpha - 1)(1 - \frac{1}{k})$ the optimal revenue that any other mechanism can get.*

■

3.2.3 Bound on the Efficiency

We next prove the bound on the efficiency of the mechanism. Our result in this subsection is a generalization of the Johari and Tsitsiklis bound [20]. Using the

assumption that there are at least k players of each type, we can improve the bound on the efficiency to $1 - \frac{1}{4k}$. In the special case when $k = 1$, we get a simple proof of the $3/4$ bound of Johari and Tsitsiklis.

To prove this bound, we first use the same argument used in [20] to assume that it is enough to consider the case where all the utility functions are linear (Lemma 3.8). For the case of linear utility functions, we use the Nash condition (3.3) to defined a separate proportional sharing game on each constraint. The bound of efficiency for these special cases can be checked easily (Lemma 3.10).

We begin with the technical lemma used in [20] to simplify the utility functions.

LEMMA 3.8 [20] *Given an instance of the game with the concave utility U_i , and let x be solution satisfying the Nash condition. Consider the game where the utility U_i is replaced by the function $W_i(z) := U'_i(x_i)z$. The allocation x still satisfies the Nash condition in the new game and the ratio between social welfare at Nash and the optimal does not increase.*

Proof. We first modify the utility function U_i to the linear function V_i with the slope $U'_i(x_i)$ such that $V_i(x_i) = U_i(x_i)$. That is $V_i(z) = U'_i(x_i)(z - x_i) + U_i(x_i)$. See figure 3.2. Because the derivative of the new utility function at x_i does not change, therefore x still satisfies the Nash condition of the new game. Furthermore, the social welfare of the solution x stays the same. On the other hand, because U_i is concave, thus $V_i(z) \geq U_i(z) \forall z$, therefore the new optimal social welfare can only increase. As a result, in the modified game, the ratio between Nash and Optimal social welfare does not increase.

Next we consider new utilities W_i obtained by shifting V_i to the origin. That

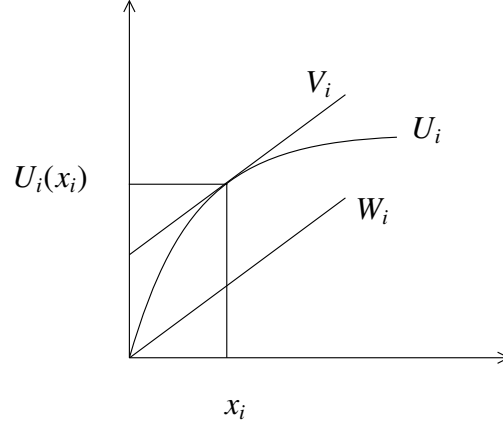


Figure 3.2: New utility functions.

is $W_i(z) = U'_i(x_i)z$. The difference between V_i and W_i is a constant. Let c be the sum of these differences over all i . If the N and O are respectively the Nash and the optimal social welfare of the game with utility V_i , then the Nash and optimal social welfare of the game with utility functions W_i are $N - c$ and $O - c$, respectively. Since we know $N \leq O$ and $0 \leq c \leq \min\{N, C\}$, we have

$$\frac{N}{O} \geq \frac{N - c}{O - c},$$

which shows that the ratio also decreases. By this we finished the proof. \blacksquare

The next lemma gives the bound on the efficiency of a simplest case.

LEMMA 3.9 *Under Assumption(α, k) the fair-sharing mechanism for the simple resource sharing problem $\sum_i x_i \leq 1$ obtains a solution with the social welfare at least $(1 - \frac{1}{4k})$ times the optimal.*

Proof. Due to Lemma 3.8, it is enough to consider the special case of pure linear utility functions $U_i(x) = a_i x$ for every players i .

The maximum social welfare is the optimum of $\sum_i a_i x_i$ where $\sum_i x_i \leq 1$. Thus it is equal to $OPT = \max_i a_i$. Let's assume that $a_1 = \max_i a_i$. From the Nash condition

$$U'_i(x_i) = a_i = \frac{p}{(1 - x_i)} \text{ if } x_i > 0,$$

one obtains the following. If $x_i > 0$, then $a_i > p$ and $a_1(1 - x_1) = p$, and thus if $x_i > 0$ then $a_i > a_1(1 - x_1)$.

By *Assumption*(α, k), in the original game there are at least k players who have the same utility function as player 1 and hence they get the same allocation x_1 by Lemma 3.4. These players provide a total utility that is at least $ka_1 x_1$ and all other players fill out the bandwidth of 1, so they have total share of $1 - kx_1$ and have utility coefficients $a_i \geq a_1(1 - x_1)$. This gives us a total utility of at least

$$\sum_i a_i x_i \geq ka_1 x_1 + a_1(1 - x_1)(1 - kx_1).$$

Hence the ratio between social welfare at the Nash equilibrium x and the optimum one is

$$\frac{\sum_i a_i x_i}{a_1} \geq \frac{ka_1 x_1 + a_1(1 - x_1)(1 - kx_1)}{a_1} = 1 - x_1 + kx_1^2.$$

This expression is minimized when $x_1 = 1/(2k)$ when the ratio is $1 - 1/(4k)$ as claimed. ■

The lemma can be easily extended to the single resource sharing problem $\sum_i \alpha_i x_i \leq 1$ where the optimal revenue is $\max_i a_i / \alpha_i$.

LEMMA 3.10 *Under Assumption(α, k) the fair-sharing mechanism for the single resource sharing problem $\sum_i \alpha_i x_i \leq 1$ obtains a solution with the social welfare at least $(1 - \frac{1}{4k})$ times the optimal.* ■

We are now ready to prove the following theorem

THEOREM 3.11 *Under Assumption(α, k) the ratio between social welfare at Nash and the optimal one is at least $1 - \frac{1}{4k}$.*

Proof. As discussed before, we will consider separate games for each resource e . Consider a Nash equilibrium x . As before we can assume without loss of generality that the utility function is linear $U_i(x) = a_i x$ for all players, as was shown in Lemma 3.8. We use $a_i = U'_i(x_i)$.

Consider the Nash condition (3.3):

$$U'_j(x_j) = \sum_e \frac{p^e \alpha_j^e}{(1 - \alpha_j^e x_j)}, \text{ if } x_j > 0 \text{ and}$$

$$U'_j(0) \leq \sum_e p^e \alpha_j^e \text{ if } x_j = 0.$$

We will consider a separate game for each resource e . In the game corresponding to resource e player i is interested in getting an allocation z_i^e with the constraint $\sum_j \alpha_j^e x_j^e \leq 1$, and a linear utility function $v_i^e x_i^e$, where $v_i^e = \frac{\alpha_i^e p^e}{(1 - \alpha_i^e x_i)}$. If we set $x_i^e = x_i$ then by (3.3) the allocation vector x^e forms a feasible allocation at equilibrium, with total utility $\sum_i v_i^e x_i^e$.

We want to apply Lemma 3.9 for each resource e . To be able to do this, we need to see that the new game also satisfies Assumption(α, k). To see why, note that players of identical type will also have identical v_i^e values (as identical players get the same allocation) and hence remain of identical type in the new game.

Now by Lemma 3.10, the social welfare of each game e at Nash is at least $(1 - \frac{1}{4k})$ times the optimal one for the edge.

Now, due to condition (3.3) of Nash equilibrium, $\sum_e v_i^e$ is equal to $U'_i(x_i)$ when $x_i > 0$, and $\sum_e v_i^e \geq U'_i(x_i)$ otherwise. Therefore for all allocations z we have

$$\sum_e \sum_i v_i^e z_i = \sum_i z_i \sum_e v_i^e \geq \sum_i U'_i(x_i) z_i.$$

Maximizing each edge separately using $\sum_i z_i v_i^e$ and summing all the maxima we get a no smaller maximum than maximizing $\sum_i U'_i(x_i) z_i$ over all feasible allocations. Therefore Lemma 3.10 implies that the overall efficiency is at least a $(1 - \frac{1}{4k})$ fraction of the maximum possible. ■

3.3 Related Literature

Proportional sharing discussed here was first studied in the context of communication networks by Kelly [23]. The explicit formulation of the Nash equilibrium is given by Johari and Tsitsiklis [20]. The efficiency result is due to Johari and Tsitsiklis [20]. The excellent survey of Johari [19] contains many references to other related works in the area.

Most of the new results in this chapter, such as the efficiency bound for the case of symmetric users and the revenue performance of proportional sharing are given by the author and Tardos in [42].

The fact that properties of some systems improve as the number of users increases has been previously considered in other settings. Edgeworth [11] considers an exchange economy, where users come to the market with a basket of goods and aim to exchange the goods to maximize their utility. He was comparing the concept of Walrasian competitive equilibrium to the notion of the core in this setting. For an exchange economy a competitive allocation is an al-

location resulting from market clearing prices p , where all players sell at price p and use the resulting money to buy their optimal set of goods. The core of the exchange economy game is an allocation of goods where no subset of users can re-contract using their initial allocation to improve at least one user's utility without decreasing the utility of any of them. It is not hard to see that all competitive allocations are in the core of the exchange game, but in general the core has other allocations that are not supported by prices. Edgeworth [11] showed that with two different players if the market contains many copies of each player, the set of core allocations converges to the competitive allocation as the number of players grows. More generally, the core in exchange economies with many (small) players is known to converge to the competitive allocations, see Anderson [4] for a survey.

CHAPTER 4

REVENUE MAXIMIZATION FOR GENERAL SINGLE-PARAMETER AUCTIONS

In the previous chapter we consider the proportional sharing mechanism for a general polyhedral setting, and prove a revenue bound of the mechanism with the symmetry assumption among competing users. The revenue in this setting was compared with the optimal social welfare. The goal of this chapter is to relax the assumption of symmetry among competing users. We will consider a special case where *all utility functions are linear* $U_i(x_i) = v_i x_i$.

The first problem we face is that, without the symmetry assumption, maximum social welfare is too strong to use as an upper bound of the revenue. To see this, consider an auction for an unit of a divisible resource, and we assume that there is a “big” user with a much higher valuation than others. In order to get a revenue that is competitive with optimal social welfare, the social welfare of the outcome has to be large also, because social welfare is an upper bound of the revenue. Thus, we need to assign most of the resource to the big user, even if he significantly misrepresents his valuation. And hence, we cannot expect to extract his valuation as income. A more rigorous argument is analyzed in [44].

In the auction for single item described above, the second highest valuation among v_i is considered as a simple and natural benchmark for the revenue. In a different setting, the case of digital good auction (e.i there is an unlimited supply of good), Goldberg et al. [12] introduce a benchmark called \mathcal{F}^2 and give a mechanism obtaining a revenue at least a constant fraction of the benchmark. Extending these benchmarks for more complex settings is an open problem.

The general setting is usually called *general single parameter auction* [15]. In this problem, each agent has a private valuation for receiving service and there is a set system representing feasible sets. See Figure 4.1. A feasible set is a set of agents that can be served simultaneously. For example in single good the feasible set system contains singletons. We focus on the typical case of downward-closed environment where every subset of a feasible set is again feasible. Another example of such an environment is a combinatorial auction with single-minded bidders, where feasible sets correspond to subsets of bidders seeking disjoint bundles of goods.

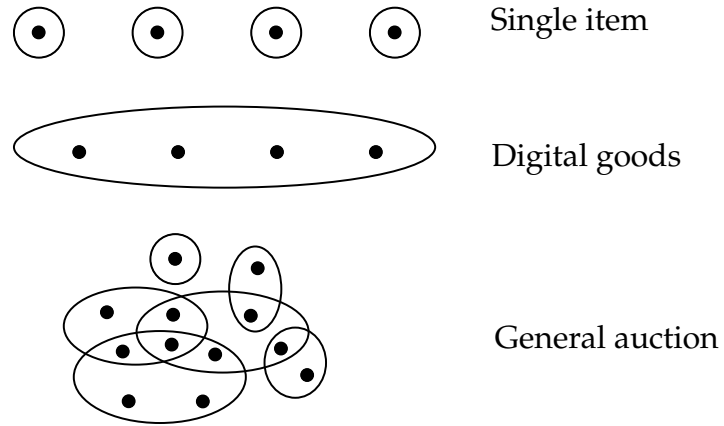


Figure 4.1: General auction setting described by a set system.

As discussed in Chapter 1, if we consider randomized outcomes in this setting, then the feasible allocation can be captured by a polyhedral. This can be stated as follows.

LEMMA 4.1 *The set of randomized outcomes of the general single parameter auction can is the set of non negative vectors $\vec{x} = (x_1, \dots, x_n)$ satisfying*

$$A\vec{x} \leq \vec{1}, \vec{x} \geq \vec{0}, \text{ where } A \text{ is a non negative matrix.}$$

We omit the proof of this claim in this chapter. A proof for a similar result can be found in the next chapter (Lemma 5.1).

The main goal of this chapter is

Define a natural revenue benchmark and design a mechanism to obtain competitive revenue against this benchmark in general auction environments.

Contribution We introduce a new revenue benchmark for the general single parameter auction and give a mechanism obtaining a revenue within a constant factor of this benchmark. Our new benchmark is the generalization of the second valuation in single item auctions and the \mathcal{F}^2 benchmark of digital good auctions. The main idea of our mechanism is to combine the general proportional sharing mechanism with the truthful mechanism for digital good of [12] to yield a competitive mechanism for the introduced benchmark.

Structure of the Chapter In Section 4.1 we formally define the benchmarks \mathcal{F}^2 and our extension for general settings. In Section 4.2 we review the auction for digital good and introduce our mechanism for the setting of an arbitrary downward-closed set system. We show that the mechanism generates at least a constant fraction of the new benchmark. Related work is discussed in Section 4.3.

4.1 Worst-case Revenue Benchmarks

The approach for prior-free mechanism design is to design an auction that is always (in worst case) within a small constant factor of some profit benchmark. We first need to define the profit benchmark we will be attempting to compete with. In the following we will discuss some worst-case benchmarks. Starting with the simplest case of auction for single item, we then discuss the benchmark for the case of digital good auction introduced in Goldberg et al. [12], and in a way of combining these two cases, we introduce a generalization of these benchmarks for the general setting of set system .

Single Item

The most simplest auction setting is the case of auctions for single item. It is well known that both first price and second price auctions give the same revenue, which is the second largest valuation. We use the second valuation because it is a simple and intuitive benchmark for single item auction.

Digital Good

For digital goods auctions, the good are in unlimited supply. One can think of natural profit benchmarks, such as (a) the maximum profit achievable with fully discriminating prices (where each bidder pays their valuation) or (b) the maximum profit achievable with a single price, are provably too strong in the sense that no truthful auction can be constant competitive with these benchmarks. Thus, the profit benchmark we will use is the following:

DEFINITION 4.1 ([12]) *The optimal single priced profit with at least two winners is*

$$\mathcal{F}^2(v) = \max_{i \geq 2} i v_{(i)},$$

where $v_{(i)}$ is the i th largest valuation.

In this chapter we also use the notation

$$\mathcal{F}(v) = \max_i i v_{(i)}.$$

General Setting

From the benchmarks of the two special cases above, we observe that in the single item auction we have a full competition among bidders, on the other hand, when the good is in unlimited supply, it is necessary to use some type of reserve price to obtain high revenue. In the general case of downward closed set system, one can think of the setting as the combination of the two extreme cases above. If one can partition the set of bidders into two set N_1 and N_2 , where N_2 is a set of bidders that can be served simultaneously, while the bidders in group N_1 need to compete with each other and with N_2 for the limited resource. From N_2 , we use the revenue benchmark of $\mathcal{F}^2(N_2)$. From N_1 we can get at most the maximum social welfare of this group of bidders, denoted by $\text{SocialOpt}(N_1)$. See Figure 4.2. For each partition, one obtains a benchmark for the revenue, thus taking the minimum over all partitions, we can define the following benchmark.

DEFINITION 4.2 (Benchmark \mathcal{R})

$$\mathcal{R} = \min_{N_1, N_2} \text{SocialOpt}(N_1) + \mathcal{F}^2(N_2),$$

where $N = N_1 \uplus N_2$ is a partition the bidders, such that N_2 can be served simultaneously.

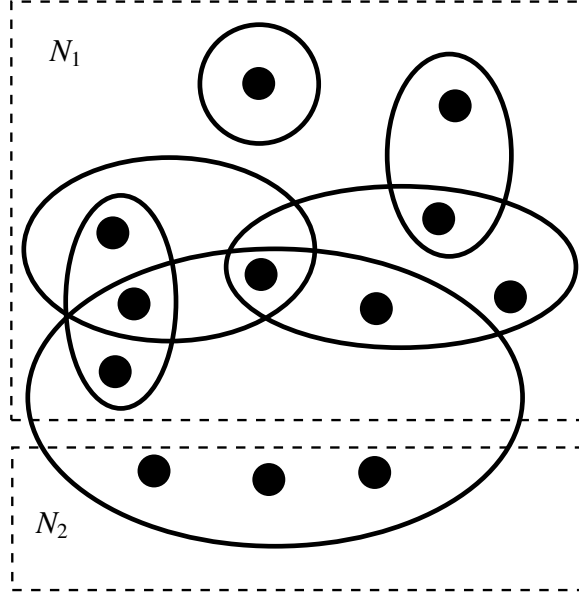


Figure 4.2: A revenue benchmark for downward closed set systems.

Remark This benchmark gives exactly the second valuation in the single item auction and the \mathcal{F}^2 benchmark in the digital good auction. Note that the revenue that we would like to obtain from N_2 is the maximal social welfare, therefore, in order to design a mechanism obtaining a constant factor of \mathcal{R} , we cannot fix the partition before running the mechanism. We will see later that the partition is part of the outcome of the mechanism.

4.2 The Mechanisms

We now discuss two mechanisms that generate a revenue at least a constant factor of the benchmarks introduced in the previous section. The first one is a truthful mechanism from [12] for the case of digital good. The second one is a combination of proportional sharing and this truthful mechanism. The second mechanism uses the solution concept of Nash equilibrium.

4.2.1 Truthful Mechanism for Digital Good Auctions

In the following, we review the auction of Goldberg et al. [12], which gives a revenue at least $1/4$ of the benchmark \mathcal{F}^2 . The main idea is to randomly partition the set of bidders into 2 groups, calculate the optimal profit can be obtained by single price in each part, and use the revenue obtained in one group as a “revenue goal” for the other group. To make this more precise, consider the following mechanism.

DEFINITION 4.3 (ProfitExtract(R) [12]) *The digital goods auction profit extractor with target profit R sells to the largest group of k bidders that can equally share R and charges each R/k .*

It is straightforward to show that ProfitExtract(R) is truthful and obtains a profit of R when $\mathcal{F}(b) = \max_i v_{(i)} \geq R$.

With this definition, we are now ready to define the following truthful mechanism, called The Random Sampling Profit Extraction auction.

DEFINITION 4.4 (RSPE [12]) *The Random Sampling Profit Extraction auction (RSPE) works as follows:*

- (i) *Randomly partition the bids b into two by flipping a fair coin for each bidder and assigning her to b' or b'' . Compute $R' = \mathcal{F}(b')$ and $R'' = \mathcal{F}(b'')$, the optimal profits for each part.*
- (ii) *Run ProfitExtract(R') on b'' and ProfitExtract(R'') on b' .*

The intuition for this auction is that $\text{ProfitExtract}(R)$ allows us treat a set of bidders, b , as one bidder with bid value $\mathcal{F}(b)$. The RSPE auction randomly partition the set of bidders and intuitively treat them as two bidders with valuations $R' = \mathcal{F}(b')$ and $R'' = \mathcal{F}(b'')$. Thus the profit of RSPE is $\min(R', R'')$. In the following we will show that this value is at least $1/4$ of the benchmark \mathcal{F}^2 .

THEOREM 4.2 ([12]) *The competitive ratio of RSPE is 4.*

Proof. As we discussed above, the profit of RSPE is $\min(R', R'')$. Thus, we just need to analyze $E[\min(R', R'')]$. Assume that $\mathcal{F}^2(b) = kp$ has with $k \geq 2$ winners at price p . Of the k winners in \mathcal{F}^2 , let k' be the number of them that are in b' and k'' the number that are in b'' . Because there are k' bidders in b' at price p , $R' \geq k'p$. Likewise, $R'' \geq k''p$. Thus,

$$\frac{E[\text{RSPE}(b)]}{\mathcal{F}^2(b)} = \frac{E[\min(R', R'')]}{kp} \geq \frac{E[\min(k'p, k''p)]}{kp} = \frac{E[\min(k', k'')]}{k} \geq \frac{1}{4}.$$

The last inequality follows from the fact that if $k \geq 2$ fair coins (corresponding to placing the winning bidders into either b' or b'') are flipped then

$$E[\min\{\#heads, \#tails\}] = \sum_{i=0}^{\lfloor k/2 \rfloor} i \cdot \binom{k}{i} \frac{1}{2^k} \geq \frac{k}{4}.$$

The equality occurs when $k = 2$. ■

4.2.2 Nash Implementation for the General Setting

We now give a mechanism whose revenue at Nash equilibrium is within a constant factor of the Benchmark \mathcal{R} .

The Main Idea In order to obtain a constant factor of the benchmark \mathcal{R} , we need to combine two important features of the mechanisms for single good and digital good described above. Recall that in the case of single item auction, we have a full competition among bidders, on the other hand, for the digital good, we need to use a version of reserve prices.

In the general case, we will first use the proportional sharing mechanism introduced in the previous chapter as a version of creating competition among bidders, after this we will give additional resources to bidders for extra money.

After the proportional sharing mechanism, we consider the bidders who get a large share of resources, which we call *big bidders*. The intuition is that there is a lack of competition among these bidders, and therefore, we can use the mechanism designed for the benchmark \mathcal{F}^2 for the big bidders.

The idea seems to be quite simple: attach a truthful mechanism after a proportional sharing mechanism. There are, however, several issues. Because the second mechanism is run on the outcome of the first mechanism, bidders might behave differently from the case where the two mechanism are run separately. There are two main issues.

First, the second phase of the mechanism is run only for the set of big bidders who get large share of the resource in the first round, therefore, it might be the case that the *small bidders* will overbid in the first round to get to the second one. Thus, the property of a Nash equilibrium in proportional sharing might be not valid. For example, the game in this case might not have a pure Nash equilibrium.

Second, it is also possible that the large bidders will either increase or lower

their original bids in the first round to change the set of bidders that survive to the second round, and thus the price of the second mechanism might be different and better for him.

To overcome the first difficulty, we modify the paying scheme in the second round of the mechanism. The price that a bidder needs to pay is the maximum of the two values: the price obtained in the second round and a price related to the price that the bidder pays in the first round. By doing this we make sure that if the small bidders overbid in the first round, they still need to pay a large money in the second round, and their payoff will be negative if he does so. We define formally the modified version of RSPE (Definition 4.4) as follow.

DEFINITION 4.5 (RSPE*(\vec{p})) *Input: a given price p_i for each bidder i . Let b_i be the bid from bidder i .*

- (i) *Randomly partition the bids b into two by flipping a fair coin for each bidder and assigning her to b' or b'' . Compute $R' = \mathcal{F}(b')$ and $R'' = \mathcal{F}(b'')$, the optimal profits for each part.*
- (ii) *Find the largest group of bidders among b' that can equally share the profit (R''), the number of these bidders is k' . Charge bidder i $\max\{\frac{R''}{k'}, p_i\}$.
Find the largest group of bidders among b'' that can equally share the profit (R'), the number of these bidders is k'' . Charge bidder i $\max\{\frac{R'}{k''}, p_i\}$.*

To overcome the second difficulty, we will slightly change the proportional sharing such that at an equilibrium, if a bidder gets a large share of the resource, then by bidding differently from the equilibrium, he cannot benefit in the second round of the mechanism.

To make it more precise, consider the simple case of sharing an unit of a single resource. We would like to modify the proportional sharing such that the following is true. Consider the set of bidders who get at least c fraction of the resource *at a Nash equilibrium*, if any of these bidders lowers his bid, he will get less than c , furthermore, he cannot change the set of the big bidders by over bidding. Note that this condition does not hold in traditional proportional sharing because by overbidding a bidder can change the price of the resource and other bidders will get less resource. To this end, we introduce the following mechanism called *Truncated Proportional Sharing* (TPS).

DEFINITION 4.6 (TPS(c)) *The Truncated Proportional Sharing mechanism is for the resource constraint $\sum_i \alpha_i x_i \leq 1$, and an upper limit c is the following.*

Each bidder i bids b_i . Let

$$b_i^* = \begin{cases} b_i & \text{if } \frac{b_i}{\alpha_i \sum_j b_j} \leq c, \\ b \text{ such that } \frac{b}{\alpha_i (b + \sum_{j \neq i} b_j)} = c & \text{if } \frac{b_i}{\alpha_i \sum_j b_j} > c. \end{cases}$$

The allocation for bidder i is

$$x_i = \min \left\{ c, \frac{b_i^*}{\alpha_i \cdot \sum_j b_j^*} \right\}.$$

The payment for bidder i is b_i .

As the name of the mechanism suggests, the Truncated Proportional Sharing mechanism above is a modified version of traditional proportional sharing, where the resource is truncated by c , and the bid b_i is also truncated by a value at which bidder i gets c fraction of the resource. Thus, we can see that at a Nash equilibrium, no bidder i bids more than b_i^* , furthermore, if he bids less than b_i^* , then $x_i < c$ and he cannot change the set of big bidders by bidding more than b_i^* .

Now because at a Nash equilibrium, $b_i = b_i^*$, to analyze the Nash equilibria, we can see this game as a proportional sharing game discussed in Chapter 3. Observe that the resource that each bidder i can get is $\min\left\{c, \frac{b_i}{\alpha_i \sum_j b_j}\right\}$. This is exactly description of the game where each bidder has two constraints $x_i \leq c$ and $\sum_i \alpha_i x_i \leq 1$. Thus with the argument above and applying the basic result of Theorem 3.1, we obtain the following.

LEMMA 4.3 *Assuming the valuation of bidder i is v_i , there is a Nash equilibrium of the mechanism TPS and the condition for the equilibria is the following*

$$b_i = b'_i \text{ for all } i, \text{ and let } p = \sum_i b_i,$$

$$\sum_i \alpha_i x_i \leq 1; \quad 0 \leq x_i \leq c \text{ for all } i,$$

$$v_i = \frac{p\alpha_i}{(1 - \alpha_i x_i)}, \text{ if } 0 < x_i < c,$$

$$v_i \leq \frac{p\alpha_i}{(1 - \alpha_i x_i)}, \text{ if } x_i = 0,$$

$$v_i \geq \frac{p\alpha_i}{(1 - \alpha_i x_i)}, \text{ if } x_i = c.$$

Furthermore, if bidder i gets c fraction of the resource, then by increasing his bid, he does not influence other bidders' strategies and by lowering his bid, he gets less than c fraction of the resource. ■

We are now ready to define our main mechanism, called Two-Phase Mechanism.

DEFINITION 4.7 (TPM(c_1, c_2)) *The Two-Phase Mechanism is for a general polyhedral environment of the form $A\vec{x} \leq \vec{1}$, each constraint (row) e of A is $\sum \alpha_i^e x_i \leq 1$. The mechanism uses the parameters c_1, c_2 , where $\frac{c_1}{2} < c_2 < c_1 < 1$. These parameters will be chosen later to optimize the revenue bound. The mechanism consists of two phases:*

- (i) Run proportional sharing for the environment $\frac{1}{c_1} \cdot A\vec{x} \leq \vec{1}$, but use the Truncated Proportional Sharing TPS(c_2) on each constraint. At Nash equilibrium, we obtain an allocation vector and a price p^e on every constraint e and the bid vector \vec{b} at Nash equilibrium.
- (ii) Let $p_i = \frac{1-c_1}{c_1-c_2} \cdot \sum_e \alpha_i^e p^e$. Run RSPE*(\vec{p}) (Definition 4.5) (but scaled down the bids by $1 - c_1$, because bidders will get at most $1 - c_1$ in this round) on the bidders that obtained c_2 in the first round.

The main result in this chapter is the following.

THEOREM 4.4 *Given an arbitrarily small ϵ , there are proper parameters c_1, c_2 such that the revenue at Nash equilibrium of the mechanism TPM with these parameter is at least $\frac{\mathcal{R}}{14+\epsilon}$, where \mathcal{R} is the benchmark defined in Definition 4.2.*

Before proving this theorem, we first derive a condition for a Nash equilibrium. As described above, we will consider the first phase of the mechanism as if there were no second round. We then claim that this condition also holds for Nash equilibrium of the extended game with the second round. The precise statement is the following.

LEMMA 4.5 *Let $0 \leq \alpha_i^e$ be the entries of the matrix A that describes the resource constraints $A\vec{x} \leq \vec{1}$. User i 's utility is $v_i x_i$. Consider the mechanism TPM in Definition 4.7. An allocation \vec{x} a bid vector \vec{b} of the first round is in a Nash solution if and only if:*

$$\begin{aligned}
\sum_i \alpha_i^e x_i &\leq c_1 && \text{for all } e \in E \\
0 \leq x_i &\leq c_2 && \text{for all } i \\
v_j &\geq \sum_e p^e \frac{\alpha_j^e / c_1}{(1 - \alpha_j^e x_j / c_1)}, && \text{if } x_j = c_2;
\end{aligned}$$

$$v_j \leq \sum_e p^e \alpha_j^e / c_1 \quad \text{if } x_j = 0;$$

$$v_j = \sum_e p^e \frac{\alpha_j^e / c_1}{(1 - \alpha_j^e x_j / c_1)}, \quad \text{if } 0 < x_j < c_2,$$

where $p^e = \sum_i b_i^e$.

Proof. It is straight forward to see that the condition above is the condition of a Nash equilibrium without the second phase of the mechanism. The proof is exactly the same as the proof of Theorem 3.1.

We now need to see that with the second round, a Nash equilibrium of the first round is still a Nash equilibrium of the extended mechanism.

For the small bidders, we will show that if he increases his bid in Nash equilibrium to get c_2 unit of resource to enter the second round the unit price that he needs to pay is larger than his valuation. Because we know that $x_i = 1$ is a feasible solution, thus $\alpha_i^e \leq 1$ for all i, e . Observe that if $0 \leq x_i < c_2$, then

$$v_i = p^e \sum_e \frac{\alpha_i^e / c_1}{(1 - \alpha_i^e x_i / c_1)} \leq p^e \sum_e \frac{\alpha_i^e / c_1}{(1 - c_2 / c_1)} = \frac{1}{c_1 - c_2} \sum_e p^e \alpha_i^e$$

Now, if bidder i increases the bids, p^e will also increase, and because in the second round, the price per unit is at least $\frac{1}{c_1 - c_2} \sum_e p^e \alpha_i^e$ for bidder i . Thus, bidder i cannot benefit from overbidding.

Because of Lemma 4.3, if a big bidder decreases his bid, he will get less than c_2 and will not be able to enter the second round and he would not increase the bid either, because by doing so, he will need to pay more, but cannot affect the strategies of any other bidders. ■

We now show a lemma which bounds the revenue obtained in the first round with the optimal social welfare of the smaller bidders.

LEMMA 4.6 *Let N_0 be the set of bidders whose valuation $v_i \leq \frac{1}{c_1 - c_2} \sum_e p^e \alpha_i^e$. Let N_1 be the set of bidders obtaining less than c_2 , then we have $N_1 \subset N_0$ and*

$$\sum_e \frac{1}{c_1 - c_2} p^e \geq \max_{z \geq 0: Az \leq 1} \sum_{i \in N_0} v_i z_i \geq \max_{z \geq 0: Az \leq 1} \sum_{i \in N_1} v_i z_i.$$

Proof. Similar to the prove above, for $0 \leq x_i < c_2$, we have

$$v_i = p^e \sum_e \frac{\alpha_j^e / c_1}{(1 - \alpha_j^e x_j / c_1)} \leq p^e \sum_e \frac{\alpha_j^e / c_1}{(1 - c_2 / c_1)} = \frac{1}{c_1 - c_2} \sum_e p^e \alpha_i^e.$$

This shows that $N_1 \subset N_0$. To show the inequality, we use the duality theorem.

$$\max_{z \geq 0} \left\{ \sum_i v_i z_i : Az \leq 1 \right\} \leq \min_{w \geq 0} \left\{ \sum_e w^e : \sum_e w^e \alpha_i^e \geq v_i \right\}.$$

Recall that α_i^e are the entries of matrix A . Applying this duality lemma, in our case $w^e = \frac{p^e}{c_1 - c_2}$. Thus, we have

$$\sum_e \frac{1}{c_1 - c_2} p^e \geq \max_{z: Az \leq 1} \sum_{i \in N_0} v_i z_i.$$

■

We are now ready to prove our main theorem.

Proof of Theorem 4.4 The mechanism in the first round gives us a partition of the bidders into N_1 and N_2 , where N_2 is the set of big bidders, who get $c_2 > c_1/2$ fraction of the resource, and N_1 is the set of the remaining bidders (small bidders). Let \vec{y} be an allocation vector of the first round and \vec{z} of the second round of the mechanism. Let R_1, R_2 be the expected revenue obtained in the first and second round, relatively.

We first show that the large bidders form a feasible set, that is, they can be served simultaneously. Recall that an allocation needs to satisfy $x_i + x_j \leq 1$,

whenever bidder i and j cannot be served together. In the first round of the mechanism, we scaled the resource down by c_1 , therefore, if an allocation vector \vec{y} satisfies $y_i + y_j > c_1$, then i, j can be served simultaneously. We choose the set N_2 to be the bidders who get $c_2 > c_1/2$, therefore, N_2 is a feasible set.

Next, we show that the final allocation vector $(\vec{y} + \vec{z})$ is feasible, that is $A(\vec{y} + \vec{z}) \leq \vec{1}$. In the first round we have that $A\vec{y} \leq c_1\vec{1}$. The second round allocates resource to the bidders in N_2 . As shown above that N_2 is a feasible set. This means that the allocation vector $\vec{1}_{N_2}$, which corresponds to servicing all the bidders in N_2 , satisfies $A\vec{1}_{N_2} \leq \vec{1}$. However, in the second round of our mechanism, we allocate to each bidder at most $1 - c_1$, therefore, $A\vec{z} \leq (1 - c_1)\vec{1}$. From this we have $A(\vec{y} + \vec{z}) \leq \vec{1}$, which we need to show.

Finally, we prove an lower bound on the revenue of our mechanism. According to Lemma 4.6, the revenue obtained in the first round is at least

$$R_1 \geq (c_1 - c_2) \max_{z: Az \leq 1} \sum_{i \in N_2} v_i z_i.$$

Therefore,

$$\frac{R_1}{c_1 - c_2} \geq \text{SocialOpt}(N_1). \quad (4.1)$$

In the second round of the mechanism we run a mechanism to subtract \mathcal{F}^2 revenue benchmark for the bidder N_2 (scaled by $1 - c_1$). Using Theorem 4.2, one would expect to have $\frac{R_2}{1 - c_1} \geq \frac{\mathcal{F}^2(N_2)}{4}$. However, the mechanism we use in the second round is slightly different from the mechanism RSPE of Definition 4.4. The bidder i 's payment for $1 - c_1$ of the resource is the maximum of the price he would need to pay in the original RSPE mechanism scaled by $1 - c_1$ and $\frac{1 - c_1}{c_1 - c_2} \cdot \sum_e \alpha_i^e p^e$. Therefore, we would get 0 revenue from bidder i with $v_i < \frac{1}{c_1 - c_2} \cdot \sum_e \alpha_i^e p^e$. However, according to Lemma 4.6, we have that $\frac{R_1}{c_1 - c_2}$ is at least the optimal social

welfare of these bidders, hence if we would have a weaker inequality as follow

$$\frac{R_1}{c_1 - c_2} + \frac{R_2}{1 - c_1} \geq \frac{\mathcal{F}^2(N_2)}{4}. \quad (4.2)$$

Thus, combining (4.1) and (4.2), we have

$$\frac{5}{c_1 - c_2} R_1 + \frac{4}{1 - c_1} R_2 \geq \text{SocialOpt}(N_1) + \mathcal{F}^2(N_2).$$

We choose $c_1 = 5/7, c_2 = 5/14 + \epsilon'$, where ϵ' is positive but negligible, one have

$$\frac{5}{5/14 - \epsilon'} R_1 + \frac{4}{2/7} R_2 \geq \text{SocialOpt}(N_1) + \mathcal{F}^2(N_2).$$

Thus for any $\epsilon > 0$, we can choose $\epsilon' > 0$ such that

$$(14 + \epsilon)(R_1 + R_2) \geq \text{SocialOpt}(N_1) + \mathcal{F}^2(N_2).$$

This is what we need to prove. ■

4.3 Related Literature

The main new result in this chapter are in [41]. The benchmark and mechanism for digital good are from the work of Goldberg et al. [12].

Profit maximization in mechanism design has an extensive history beginning, primarily, with the seminal paper of Myerson [40] and similar results by Riley and Samuelson [50]. These papers study Bayesian optimal mechanism design. This material is by now standard and can be found in basic texts on auction theory [37, 25].

Worst-case benchmark approach was first introduced by Goldberg et al. [12], and is followed by many others. This type of mechanism is now commonly

called prior-free mechanism design. For more details on this line of work see the recent papers [16, 15] and the survey [14].

Mechanism design via Nash equilibrium implementation in full information settings has a large literature in economics, started with the seminal paper of Maskin [33], see the recent survey [34, 48] and the citations therein.

Proportional sharing mechanism and its extension for networks is introduced and studied in [24, 18, 20, 21]. Most of these works study the social welfare of the system. The work presented in this thesis and the related papers [42, 43, 44, 41] study the revenue of this class of mechanisms.

CHAPTER 5

WEIGHTED PROPORTIONAL SHARING AND KEYWORD AUCTIONS

In this chapter we study the weighted proportional sharing mechanism. This is a natural extension of the classical proportional sharing studied in Chapter 3, that provides a framework to analyze the incentives of revenue maximizing providers. In weighted proportional sharing, providers will decide different weights on each user. This type of mechanisms in principle is similar to the theory of price discrimination in economics literatures. To motivate our questions in this chapter, we first explain the price discrimination frameworks in traditional Bayesian mechanism design and in sponsored search applications.

Price Discrimination Price discrimination [56, 57], or price differentiation, exists when sales of identical goods or services are transacted at different prices from the same provider. In communication and information technology markets different types of discrimination pricing is critical for sellers [54, 7]. Price discrimination has been considered in many settings. The case of full information games is considered by Varian [59, 60], that assumes that each market has a demand function and the monopoly charges different unit prices on each market to maximize the revenue. The question considered by Varian is the effect of this discrimination pricing scheme on the social welfare.

Price Discrimination in Sponsored Search In sponsored search auctions, the winning bidders are not the firms with the highest per-click bids: advertisers are ranked on the basis of the product of their bid and a factor that is something like an estimated click-through rate. The rough motivation for this is straight-

forward: weighting bids by their click-through rates is akin to ranking them on their contributions to search-engine revenues (as opposed to per-click revenues which is a less natural objective) [28, 6, 58, 10]. However, there are two issues. First, it is not clear that one can use a single scale as in click-through rates model to capture that ad's quality. This is because other ads that appear on the same page can affect an ad's quality (called externalities). Second, because the click-through rates are given by the search engines. One wonders if these weights are affected by strategic actions of auctioneer to maximize the revenue in the same principle as the discrimination pricing framework.

Our Question Motivated by this line of research, in this chapter we study a *simple version of price discrimination for proportional sharing*. We introduce the weighted proportional sharing mechanism, which is a natural generalization of the traditional proportional sharing mechanism studied in Chapter 3. In this mechanism, each user i is allocated

$$x_i = \frac{w_i}{\sum_j w_j} C_i,$$

where C_i is a value decided by strategic providers. This new class mechanisms is simple to describe to users, and more suitable for resources with general convex constraints as users use the resources at different rates. Our question is

When provider uses the weighted proportional sharing mechanism to discriminate among users, how much revenue can the provider get and what is the efficiency loss?

Our Results The results in this chapter can be summarized as follows.

Revenue: The revenue of the weighted proportional allocation is at least $k/(k+1)$ times the revenue under standard third-degree price-discrimination with a set of k users excluded. In a third-degree discrimination pricing scheme, a provider can impose different unit prices for different users to maximize the revenue. The maximum revenue in this case is $\max \sum_i U'_i(x_i)x_i$ over $\vec{x} \in \mathcal{P}$. Compared with this benchmark, the traditional proportional sharing mechanism can have arbitrarily small revenue. We also note that comparing revenue of a mechanism to a maximum revenue obtained by other optimal pricing scheme where some users do not participate has been widely used in the mechanism design literature, see for example [14]. Here we establish a general theorem in this line of work on revenue maximizing mechanisms.

Efficiency: For linear user utility functions, the social welfare is at least $1/(1 + 2/\sqrt{3}) \approx 0.464$ times the maximum social welfare, and this bound is tight. We extend this result by introducing a broad class of utility functions, we call δ -utility functions (δ is a non-negative parameter), and show that this class of utility functions contains many families of utility functions found in literature. For example, a linear or truncated linear utility function as well as $\log(1+x)$, some polynomials and some families of utility functions commonly considered in the network resource allocation belongs to this class with $\delta \leq 2$. We also show that a utility function from this class remains in the class by multiplying with any positive constants and that sums of utility functions from this class remain in the class. We then show that if the utility functions are δ -utility functions, then the social welfare is at least $1/(1 + 2/\sqrt{3} + \delta)$ times the maximum social welfare.

Organization of the Chapter We first discuss the sponsored search auction used and some of the drawbacks of the current model in Section 5.1. We then describe the weighted proportional sharing game in an arbitrary polyhedron as an alternative approach for this problem in Section 5.2. The revenue gain, the price of anarchy of this mechanism and the environment with many providers will be studied in the Section 5.3, 5.4 and 5.5 relatively. The related literature is discussed in Section 5.6.

5.1 Sponsored Search Applications

In the keyword auctions for sponsored search, the general second price (GSP) mechanism is in common use by search engines. GSP is an algorithm for placing ads to ad-slots, where the bids of advertisers are multiplied by weights that can be different for different advertisers and such weighted bids are used for placing the ads. The larger the weighted bid, the better the position that the ad gets. The reason to introduce these weights is explained by the term click-through rates r_{ij} , which is the probability that users click on ad i when it is placed at position j . It is commonly assumed that $r_{ij} = \alpha_i \times \beta_j$, where α_i is the quality of an ad i capturing how relevant the ad is to the search keyword, and β_j is the quality of the position: a large value of β_j is associated to a good position among the sponsored links.

This commonly used approach has two important issues. First, it is possible that in addition to the ad's quality, other ads that appear on the same page can affect its click-through rate (externalities). Second, it is not clear what the real connection is between the weights assigned to each advertiser and the ads'

quality α_i , as the search engine might strategically assigns these weights to maximize the revenue.

Externalities in keyword auctions are natural and important, for example, the valuation of a bidder for being, say, in position 2 depends on what ad is showed in position 1. For example, NIKE in position 1 makes position 2 less valuable for *sneakers* compared to having an unknown brand name in position 1.

Polyhedral Environments for Keyword Auctions

The general polyhedral framework can be used to capture a very general dependency among click-through rates. The model can be described as follow.

The auction is for a single keyword, and there are n advertisers bidding to have their ad appear as a sponsored search result. There are finite set of outcomes, depending on which bidder gets displayed in which position. We describe each of these outcomes as a n dimensional vector whose coordinates are the expected number of clicks that the corresponding advertiser gets. More precisely, let $\vec{x}^1, \dots, \vec{x}^N$ be all the possible outcome vectors, and $\vec{x}^k = (x_1^k, \dots, x_n^k)$, where x_i^k is the expected number of clicks that advertiser i receives at outcome k . To think of keyword auction as a convex resource allocation, we need to allow randomization in the allocation of bidders to positions. Choosing between the deterministic allocations by the probability distribution $\vec{p} = (p_1, \dots, p_N)$, we have that $\sum_j p_j \vec{x}^j$ is the vector whose coordinates correspond to the expected number of clicks of an advertiser. Now the set of expected allocation vectors

obtained this way is exactly the convex hull

$$\text{conv}(\vec{x}^1, \dots, \vec{x}^N) = \{\vec{x} : \vec{x} = \sum_j p_j \vec{x}^j, p_j \in [0, 1] \text{ for every } j \text{ and } \sum_j p_j = 1\}.$$

In this model, we will assume a natural condition on the externalities of the click-through rates: if we remove an ad from a position (by simply showing one fewer ad), the expected number of clicks received by the remaining ads does not decrease. Under this assumption, it is not hard to see that the set of all possible randomized allocation vectors, that is the convex hull $\text{conv}(\vec{x}^1, \dots, \vec{x}^N)$, can be written as $\{\vec{x} : A\vec{x} \leq \vec{1}, \text{ where } A \geq 0\}$, which is exactly the constraint of the problem considered in this thesis. This statement can be formulated as follow.

We say that the a resource allocation problem with a feasible set of allocations \mathcal{P} in \mathbb{R}_+^n satisfies the *assumption of non-positive externalities* if for any allocation $\vec{x} \in \mathcal{P}$, and any coordinate k , there is an allocation $\vec{x}' \in \mathcal{P}$ such that (1) $x'_k = 0$, and (2) $x_i = 0$ implies $x'_i = 0$, and $x'_i \geq x_i$ for all $i \neq k$.

LEMMA 5.1 *The convex hull of some non-negative vectors $\vec{x}^1, \dots, \vec{x}^N$ that satisfy the assumption of non-positive externalities can be written as $\{\vec{x} \in \mathbb{R}_+^n : A\vec{x} \leq \vec{1}\}$, for a non negative matrix A .*

Proof. Let C be the convex hull of $\vec{x}^1, \dots, \vec{x}^N$. We will show that if these vectors satisfy the assumption of non-positive externalities, then for a vector $\vec{w} \in C$, and any vector \vec{v} such that $0 \leq \vec{v} \leq \vec{w}$ is also in C . With this property, it is not difficult to see from basic convex geometry that the set C can be written as a polyhedron.

We prove this property by induction on the number of non-zero coordinates of \vec{w} . The claim is trivial when $w_i = 0$ for every i . Consider a vector $\vec{w} \in C$.

The set C is the convex hull of $\vec{x}^1, \dots, \vec{x}^N$, hence there exist non-negative real numbers $\alpha_1, \dots, \alpha_N$ such that $\sum_i \alpha_i = 1$, and $\vec{w} = \sum_i \alpha_i \vec{x}^i$. Let k be the coordinate that minimizes the ratio v_i/w_i for $w_i \neq 0$, and let λ denote this ratio. By definition, $\lambda \leq 1$, and thus if $\lambda = 1$, there is nothing to prove. We use the definition of non-positive externalities for each \vec{x}^i to obtain a vector $\vec{z}^i \in C$ with $z_k^i = 0$, and let $\vec{w}' = \sum_i \alpha_i \vec{z}^i$. Consider the vector $\lambda \vec{w} + (1 - \lambda) \vec{w}' \in C$. By definition of λ , we have the following facts

- $\vec{w}' \in C$;
- \vec{w}' has more zero coordinates than \vec{w} (namely the k th coordinate);
- the k th coordinate of $\lambda \vec{w} + (1 - \lambda) \vec{w}'$ is equal to v_k ;
- $\lambda \vec{w} + (1 - \lambda) \vec{w}' \geq \vec{v}$ (as $\vec{w}'_j \geq \vec{w}_j$ for all coordinates $j \neq k$);
- $\lambda \vec{w} \leq \vec{v}$ which follows as λ was the minimum ratio $\min_i v_i/w_i$.

The last two properties guarantee that there is a vector $0 \leq \vec{v}' \leq \vec{w}'$ such that $\lambda \vec{w} + (1 - \lambda) \vec{v}' = \vec{v}$. Now, we use the induction hypothesis to $\vec{w}' \in C$ and $0 \leq \vec{v}' \leq \vec{w}'$ to show that $\vec{v}' \in C$, and hence, $\vec{v} = \lambda \vec{w} + (1 - \lambda) \vec{v}' \in C$. ■

5.2 Weighted Proportional Mechanism

The weighted proportional allocation game is defined as a two-stage Stackelberg game as follows. The provider first announces a set of discrimination weights $\vec{C} = (C_1, \dots, C_n)$, then users submit bids $\vec{w} = (w_1, \dots, w_n)$. The allocation to each user i is given by $x_i = C_i w_i / \sum_j w_j$. In this game, each user i chooses a bid w_i that maximizes the surplus $U_i(x_i) - w_i$. The provider, however, could

predict the Nash outcome, which will be shown to be unique, and he selects the discrimination weights \vec{C} that maximize the revenue $R = \sum_j w_j$ with the condition that the resulting Nash allocation vector is feasible, i.e. $\vec{x} \in \mathcal{P}$. Note that this condition is only required for the allocation at Nash equilibrium.

Given a discrimination weight C_i and the sum of the bids $\sum_j w_j$, each user i selects a bid w_i that maximizes his surplus, i.e. solves

$$\text{USER: maximize } U_i\left(\frac{w_i}{\sum_{j \neq i} w_j + w_i} C_i\right) - w_i \text{ over } w_i \geq 0. \quad (5.1)$$

Under the assumed behavior of users, one can analyze the Nash equilibrium of the game. It turns out that a Nash equilibrium exists and is unique, and at Nash equilibrium the relation between revenue and allocation is captured by an implicit function. This is stated in the following lemma.

LEMMA 5.2 *Given discrimination weights \vec{C} , there is a unique allocation corresponding to the unique Nash equilibrium. Vice versa, given an allocation \vec{x} , there is a weight \vec{C} such that \vec{x} is the corresponding outcome. Furthermore, the corresponding revenue $R(\vec{x})$ is a function of \vec{x} given by*

$$\sum_i \frac{U'_i(x_i)x_i}{U'_i(x_i)x_i + R(\vec{x})} = 1. \quad (5.2)$$

Proof. We have

$$x_i = C_i \frac{w_i}{\sum_j w_j} \quad (5.3)$$

and the user's problem can be written as

$$\text{USER: maximize } U_i\left(\frac{w_i}{\sum_{j \neq i} w_j + w_i} C_i\right) - w_i \text{ over } w_i \geq 0. \quad (5.4)$$

Note that the objective function in (5.4) is concave in w_i , hence, at an optimum solution either $w_i = 0$ or the derivative of the objective function is zero. Setting

the derivative to zero is equivalent to:

$$U'_i(x_i) \cdot C_i \frac{\sum_{j \neq i} w_j}{(\sum_j w_j)^2} = 1, \text{ for } x_i > 0.$$

It follows

$$U'_i(x_i) = \frac{(\sum_j w_j)^2}{C_i \sum_{j \neq i} w_j} = \frac{R^2}{C_i(R - w_i)} \quad (5.5)$$

where recall the revenue is equal to the sum of the payments by individual users, i.e. $R = \sum_j w_j$. Combining with $w_i = x_i R / C_i$ that follows from (5.3), we have

$$U'_i(x_i) = \frac{R}{C_i - x_i} \Leftrightarrow C_i U'_i(x_i) \left(1 - \frac{x_i}{C_i}\right) = R. \quad (5.6)$$

Now, $\sum \frac{x_i}{C_i} = 1$, thus, condition (5.6) is exactly the condition for maximizing

$$\sum_i \int_0^{x_i} C_i U'_i(t_i) \left(1 - \frac{t_i}{C_i}\right) dt_i \text{ over } x_i \geq 0, \text{ subject to } \sum \frac{x_i}{C_i} = 1.$$

Since $\int_0^{x_i} C_i U'_i(t_i) \left(1 - \frac{t_i}{C_i}\right) dt_i$ is a strictly concave function with respect to x_i , there exists a unique Nash equilibrium.

It remains to show that for an equilibrium allocation \vec{x} , the revenue R is given by

$$\sum_i \frac{U'_i(x_i)x_i}{U'_i(x_i)x_i + R} = 1. \quad (5.7)$$

From (5.6), we have

$$U'_i(x_i) = \frac{R}{C_i - x_i} = \frac{R}{x_i(C_i/x_i - 1)} \Rightarrow \frac{C_i}{x_i} - 1 = \frac{R}{U'_i(x_i)x_i} \Rightarrow \frac{x_i}{C_i} = \frac{U'_i(x_i)x_i}{U'_i(x_i)x_i + R}.$$

Combining with $\sum_i x_i / C_i = 1$ which follows from (5.3), we obtain (5.7). Note that all the formulas above are applied for the case $x_i > 0$ only; nevertheless, if $x_i = 0$, we have $U'_i(x_i)x_i = 0$, and therefore, the equation (5.7) holds for any optimum allocation vector \vec{x} .

Lastly, to show given an allocation \vec{x} , there is a weight vector \vec{C} , such that \vec{x} is the corresponding outcome, we note that in equilibrium, discrimination weights \vec{C} and bids \vec{w} are functions of the equilibrium allocation \vec{x} given in the following, for every i ,

$$C_i = x_i + \frac{R(\vec{x})}{U'_i(x_i)} \text{ and } w_i = \frac{R(\vec{x})}{U'_i(x_i)x_i + R(\vec{x})} U'_i(x_i)x_i.$$

By this we finished the proof of the lemma. ■

The revenue maximizing problem of the provider using a weighted proportional sharing can be written as follows

$$\text{PROVIDER: maximize } R(\vec{x}) \text{ over } \vec{x} \in \mathcal{P}, \text{ where } R(\vec{x}) \text{ is given by (5.2).} \quad (5.8)$$

5.3 Revenue

In this section, we prove a guarantee on the revenue obtained by our mechanism. Consider the standard third-degree price discrimination scheme [56, 57] where the provider can impose different unit prices for different users. If p_i is the unit price for user i , then the user's pay-off maximization problem is $\max U_i(x_i) - p_i x_i$ over $x_i \geq 0$. Thus, $U'_i(x_i) = p_i$, and therefore, the total revenue maximization problem of the provider is $\max \sum_i U'_i(x_i)x_i$ over $\vec{x} \in \mathcal{P}$. However, in revenue maximization mechanism design, comparing with such a benchmark is too ambitious. Instead, we will use the optimal revenue of this scheme in a system where some users do not participate. Namely, let R_{n-k}^* be the optimum revenue obtained by the third-degree price discrimination scheme when an arbitrary set of k users is excluded. More formally,

$$R_{n-k}^* = \min_{S \subset \{1, \dots, n\}: |S|=n-k} \max_{\vec{x} \in \mathcal{P}} \sum_{i \in S} U'_i(x_i)x_i.$$

Our main result in this section is the following theorem.

THEOREM 5.3 *Suppose that for each i , $U'_i(x)x$ is a concave function. Let R be the optimum revenue of the weighted proportional allocation mechanism, then for all $1 \leq k < n$,*

$$R \geq \frac{k}{k+1} R_{n-k}^*.$$

Proof. Let $R(\vec{x})$ be the value of R satisfying (5.2). From (5.2) it is easy to get the following

$$\sum_i U'_i(x_i)x_i - \max_j U'_j(x_j)x_j \leq R(\vec{x}) < \sum_i U'_i(x_i)x_i, \text{ for all } \vec{x} \in \mathcal{P}.$$

Suppose that for each $1 \leq k < n$, there exists $\vec{x} \in \mathcal{P}$ such that both of the following two conditions hold

- (i) $\sum_i U'_i(x_i)x_i \geq R_{n-k}^*$
- (ii) $U'_1(x_1)x_1 = \dots = U'_{k+1}(x_{k+1})x_{k+1} \geq \dots \geq U'_n(x_n)x_n$,

where, without loss of generality, the users are enumerated such that $U'_1(x_1)x_1 \geq \dots \geq U'_n(x_n)x_n$. Under conditions (i) and (ii), the theorem is followed because

$$R(\vec{x}) \geq \sum_i U'_i(x_i)x_i - \max_j U'_j(x_j)x_j \geq \frac{k}{k+1} \sum_i U'_i(x_i)x_i \geq \frac{k}{k+1} R_{n-k}^*.$$

We show that such \vec{x} exists by induction over k . *Base step:* $k = 0$. The vector \vec{x} that maximizes $\sum_i U'_i(x_i)x_i$ over $\vec{x} \in \mathcal{P}$ satisfies both conditions.

Induction step: Let $\vec{x} \in \mathcal{P}$ be a vector such that both condition (i) and condition (ii) hold for k . We then show that there exists another vector in \mathcal{P} such that these conditions hold for $k+1$. Note that $R_{n-k}^* \geq R_{n-(k+1)}^*$ as allowing to exclude a

larger set of users cannot increase R_{n-k}^* . In the following, without loss of generality, we assume that users are enumerated such that $U'_1(x_1)x_1 \geq \dots \geq U'_n(x_n)x_n$. Let $\vec{y} \in \mathcal{P}$ be an optimum solution of the provider's problem under price taking users and the constraint $y_1 = \dots = y_{k+1} = 0$, i.e. with users $1, 2, \dots, k+1$ excluded. We have that $\sum_i U'_i(y_i)y_i \geq R_{n-(k+1)}^*$.

Now, let us consider the vector $\vec{v}(t)$ defined by

$$\vec{v}(t) = (1-t) \cdot (U'_1(x_1)x_1, \dots, U'_n(x_n)x_n) + t \cdot (U'_1(y_1)y_1, \dots, U'_n(y_n)y_n), \text{ for } t \in [0, 1].$$

Note that as t increases from 0, the $k+1$ largest coordinates of $\vec{v}(t)$ decrease, while all the other coordinates either increase or do not change. Thus, there exists $t^* \in [0, 1]$ such that the largest $k+2$ coordinates of $\vec{v}(t)$ are equal. Furthermore, as $\sum_i U'_i(x_i)x_i \geq R_{n-(k+1)}^*$ and $\sum_i U'_i(y_i)y_i \geq R_{n-(k+1)}^*$, we have that $\sum_i v_i(t^*) \geq R_{n-(k+1)}^*$. Finally, since for each i , $U'_i(x_i)x_i$ is concave, there exists a vector $\vec{z} \in \mathcal{P}$ such that $(U'_1(z_1)z_1, \dots, U'_n(z_n)z_n) = \vec{v}(t^*)$. By this, we showed that the vector \vec{z} satisfies conditions (i) and (ii) for $k+1$ which completes the proof. ■

5.4 Price of Anarchy

In this section, we analyze the efficiency of the system for the case of single provider and linear user utility functions. This provides us with basic techniques for a more general result established in the next section. In particular, we prove the following theorem.

THEOREM 5.4 *Assume that for each user i , the utility function is linear, $U_i(x) = v_i x$, for some $v_i > 0$. Then, the worst-case efficiency is $1/(1 + 2/\sqrt{3})$ (approx. 46%). Furthermore, this bound is tight.*

Before proving the theorem, note that the worst-case efficiency can be achieved asymptotically as the number of users n tends to infinity. One example is when we have the resource constraint $\sum_i x_i \leq 1$, and there is a unique user with largest marginal utility, say this is user 1, all other users have identical marginal utilities equal to $(2 - \sqrt{3})^2 v_1 \approx 0.0718 v_1$. At the Nash equilibrium, user 1 obtains 42.26% of the resource and the rest is equally shared by the remaining users. Thus, the efficiency loss occurs only when there is an unbalance in the marginal utilities by the users. One can actually show that when there is a higher competitiveness among the users, the efficiency increases. More precisely, in Theorem 5.6, we show that if there are at least k users with the largest marginal utilities, then the efficiency is at least $1 - \frac{1}{2k} + o(1/k)$.

Recall that $R(\vec{x})$ is the function given by (5.2). Let R^* be the optimum revenue, i.e. $R^* = \max\{R(\vec{x}) : \vec{x} \in \mathcal{P}\}$. We have the following observation.

LEMMA 5.5 *The set $\mathcal{L}_\mu := \{\vec{x} \in \mathbb{R}_+^n : R(\vec{x}) \geq \mu\}$ is convex, for every $\mu \in [0, R^*]$.*

Proof. We want to show that $\mathcal{L}_\mu := \{\vec{x} \in \mathbb{R}_+^n : R(\vec{x}) \geq \mu\}$ is convex, where

$$\sum_i \frac{v_i x_i}{v_i x_i + R(\vec{x})} = 1.$$

It is clear that $R(\vec{x})$ is a monotone increasing function in each x_i , therefore if $\vec{y} \geq \vec{x}$, and $\vec{x} \in \mathcal{L}_\mu$, then also $\vec{y} \in \mathcal{L}_\mu$.

It is enough to see that given \vec{x} and \vec{y} such that $R(\vec{x}) = R(\vec{y}) = \mu$ then for every other vector \vec{z} on the interval connecting \vec{x} and \vec{y} we have $R(\vec{z}) > \mu$. See Figure 5.1. Since $R(\vec{z})$ is a monotone function in each z_i , it is enough to prove that

$$\sum_i \frac{v_i z_i}{v_i z_i + \mu} \geq 1.$$

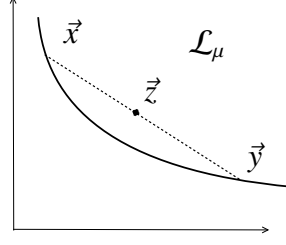


Figure 5.1: Convexity of the revenue.

Assume $\vec{z} = \alpha \vec{x} + (1 - \alpha) \vec{y}$. Since the function $\frac{v_i z_i}{v_i z_i + \mu}$ is concave for every i , we have

$$\frac{v_i z_i}{v_i z_i + \mu} \geq \alpha \frac{v_i x_i}{v_i x_i + \mu} + (1 - \alpha) \frac{v_i y_i}{v_i y_i + \mu}.$$

Summing over i , we obtain the desired inequality. ■

We now give the proof for Theorem 5.4.

Proof of Theorem 5.4. The example showing the bound is tight is given in the remark above; we now prove that the efficiency is at least $1/(1 + 2/\sqrt{3})$.

Since for every $\vec{x} \in \mathcal{P}$, $R(\vec{x}) \leq R^*$, the two convex sets \mathcal{L}_{R^*} and \mathcal{P} do not have common interior points. Let H be a hyperplane that weakly separates these two sets. This hyperplane can be written as

$$\sum_i \gamma_i x_i = 1, \text{ with } \gamma_i \geq 0 \text{ for each } i. \quad (5.9)$$

Consider the game where the provider has the feasible set $Q = \{\vec{x} \in \mathbb{R}_+^n : \sum_i \gamma_i x_i \leq 1\}$, then the allocation that maximizes the revenue over Q is the same as in the original game. Since $\mathcal{P} \subset Q$, the optimal social welfare of the new game is at least the social welfare of the original game. Therefore, it is enough to prove a lower bound on the efficiency for the class of games where the provider has the feasible set Q . See Figure 5.2. The observation above allows us to reduce

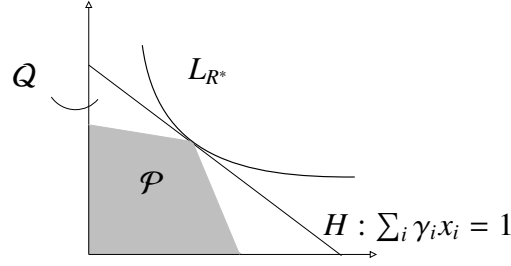


Figure 5.2: Reduction to a simple constraint.

the analysis to simpler optimization problems. In particular, the optimal social welfare in this new game is $\max_i v_i/\gamma_i$; the condition for Nash equilibrium, as argued above, is the condition for the optimal point of $R(\vec{x})$ over $\sum_i \gamma_i x_i = 1$, for which we can derive to a simple form as follow. Taking the partial derivative with respect to x_j on both sides in (5.2), with $U_i(x_i) = v_i x_i$, we have

$$\frac{\partial}{\partial x_j} \sum_i \frac{v_i x_i}{v_i x_i + R} = 0 \Leftrightarrow \frac{\partial}{\partial x_j} \frac{v_j x_j}{v_j x_j + R} - \sum_{i \neq j} \frac{\partial R}{\partial x_j} \frac{v_j x_i}{(v_i x_i + R)^2} = 0.$$

Noting that

$$\frac{\partial}{\partial x_j} \frac{v_j x_j}{v_j x_j + R} = \frac{R v_j}{(v_j x_j + R)^2} - \frac{\partial R}{\partial x_j} \frac{v_j x_j}{(v_j x_j + R)^2},$$

Thus, we have

$$\frac{R v_j}{(v_j x_j + R)^2} = \frac{\partial R}{\partial x_j} \cdot \sum_i \frac{v_i x_i}{(v_i x_i + R)^2}.$$

Now, because $R(\vec{x})$ achieves the optimum value R^* over the set $\{\vec{x} \in \mathbb{R}_+^n : \sum_i \gamma_i x_i \leq 1\}$, we have either $x_j = 0$ or $\frac{\partial}{\partial x_j} R = \lambda \gamma_j$ where $\lambda \geq 0$ is a parameter (the Lagrange multiplier associated to the constraint $\sum_i \gamma_i x_i \leq 1$). It follows that

$$\text{either } x_i = 0 \text{ or } \frac{v_i/\gamma_i}{(v_i x_i + R^*)^2} = \frac{\lambda}{R^*} \sum_i \frac{v_i x_i}{(v_i x_i + R^*)^2} = p > 0. \quad (5.10)$$

By this, we obtain a condition that at the Nash equilibrium allocation $\frac{v_i/\gamma_i}{(v_i x_i + R^*)^2}$ are equal to a common value p . Therefore, if v_i/γ_i is large then the denominator $(v_i x_i + R^*)^2$ needs to be large as well. At the same time, the optimal solution of

social welfare distributes all the resource to the user with the highest v_i/γ_i . This is the intuition for the fact that the efficiency is bounded by a constant.

First we will rescale the variables to make the equations easier to follow. We will use a new set of variables, namely $z_i = \gamma_i x_i$, and $a_i = v_i/\gamma_i$. One way to think about this new variables is to think of another game where the resource constraint is $\sum_i z_i = 1$ and user i 's utility is $a_i z_i$. Without loss of generality, we assume that $a_1 = \max_i a_i$. The optimal social welfare is $\text{OPT} = \max_{\vec{x}: \sum_i \gamma_i x_i = 1} \sum_i v_i x_i = \max_{\vec{z}: \sum_i z_i = 1} \sum_i a_i z_i = a_1$.

We now introduce new variables $y_i = v_i x_i / (v_i x_i + R^*) = a_i z_i / (a_i z_i + R^*)$. Because of (5.2), we have $\sum_i y_i = 1$. The goal of introducing these variables is to bound the optimal social welfare and the social welfare of a Nash equilibrium as functions of y_i . Now, from $y_i = a_i z_i / (a_i z_i + R^*)$, we have

$$a_i z_i = R^* \frac{y_i}{1 - y_i} \text{ and } z_i = R^* \frac{y_i}{a_i(1 - y_i)}.$$

Next, we are going to bound the social welfare of a Nash equilibrium and the optimal solution.

NASH (the social welfare of a Nash equilibrium) can be bounded using the relations above as follow

$$\begin{aligned} \text{NASH} &= \sum_i a_i z_i = R^* \sum_i \frac{y_i}{1 - y_i} \geq R^* \left(\frac{y_1}{1 - y_1} + \sum_{i \geq 2} y_i \right) \\ &= R^* \left(\frac{y_1}{1 - y_1} + (1 - y_1) \right) = R^* \frac{y_1^2 - y_1 + 1}{1 - y_1}. \end{aligned} \tag{5.11}$$

The above inequality uses the fact that $\frac{y_i}{1 - y_i} \geq y_i$ and $\sum_i y_i = 1$.

OPT (the optimal social welfare), as argued above, is $\max_i a_i = a_1$. To bound a_1 as a function of y_i , we multiply a_1 with $\sum_i z_i$, which is 1, and use the relation between

z_i and y_i to have OPT as a function of y_i . More precisely,

$$OPT = a_1 = a_1 \left(\sum_i z_i \right) = a_1 R^* \sum_i \frac{y_i}{a_i(1 - y_i)}. \quad (5.12)$$

Now, we use the condition for Nash equilibrium. (Note that this is the only place in the proof that uses (5.10).) First we rewrite the condition for the variables z_i, a_i . Replacing $a_i = v_i/\gamma_i$ and $v_i x_i = a_i z_i = R^* \frac{y_i}{1 - y_i}$ in to the condition for Nash equilibrium (5.10), we can derive

$$\text{either } y_i = 0 \text{ or } \frac{a_i(1 - y_i)^2}{R^{*2}} = p > 0.$$

From this condition, we have $a_i(1 - y_i)^2 = a_1(1 - y_1)^2$ whenever $y_1, y_i > 0$, hence $a_i(1 - y_i) = \frac{a_1(1 - y_1)^2}{1 - y_i}$. Replacing this equality in the optimal social welfare (5.12), we have

$$OPT = a_1 R^* \sum_i \frac{y_i}{a_i(1 - y_i)} = \frac{R^*}{(1 - y_1)^2} \sum_i y_i(1 - y_i) \leq \frac{R^*}{(1 - y_1)^2} \left(y_1(1 - y_1) + \sum_{i \geq 2} y_i \right).$$

The last inequality uses the fact that $y_i(1 - y_i) \leq y_i$. Using this and replacing $\sum_{i \geq 2} y_i = 1 - y_1$, we obtain

$$OPT \leq \frac{R^*}{(1 - y_1)^2} (y_1(1 - y_1) + 1 - y_1) = R^* \frac{1 - y_1^2}{(1 - y_1)^2}. \quad (5.13)$$

From (5.11) and (5.13), the efficiency is at least $(y_1^2 - y_1 + 1)/(y_1 + 1)$. By a simple calculus, one can show that this ratio is at least $1/(1 + 2/\sqrt{3})$, which is what we need to prove. ■

THEOREM 5.6 *Admit same setting as in Theorem 5.4 and, in addition, assume that at Nash equilibrium the largest users get at most $1/k$ unit of resource. Then, the efficiency is at least $1 - \frac{1}{2k} + o(1/k)$.*

Proof. Following the same steps as in the proof of Theorem 5.4, we have that the social welfare at the Nash equilibrium is at least

$$\frac{ky}{1-y} + (1-ky)$$

and the maximum social welfare is at most

$$\frac{1}{(1-y)^2}(ky(1-y) + 1-ky)$$

for some $0 \leq y \leq 1/k$. It follows that the efficiency is at least

$$f_k(y) = (1-y) \left(\frac{2-y}{1-ky^2} - 1 \right)$$

for some $0 \leq y \leq 1/k$. It remains only to establish that

$$\inf_{y \in [0, 1/k]} f_k(y) = 1 - \frac{1}{2k} + o(1/k).$$

This follows by noting that for a minimizer y , $f'_k(y) = 0$, which is equivalent to

$$y^4 - \frac{5}{k}y^2 + \frac{2}{k} \left(2 + \frac{1}{k} \right) y - \frac{2}{k^2} = 0.$$

Since $y \leq 1/k$, we neglect the term y^4 as it is of smaller order than other terms, which amounts to solving a quadratic equation whose solution in $[0, 1/k]$ is given by

$$y = \frac{1}{5} \left(2 + \frac{1}{k} - \sqrt{4 - \frac{6}{k} + \frac{1}{k^2}} \right).$$

It readily follows that $y = \frac{1}{2k} + o(1/k)$ and plugging into $f_k(y)$ yields the asserted claim. ■

5.5 Multiple Providers

In this section we will extend the results to the case of multiple competing providers, and a broad class of utility functions. We first define the framework for multiple providers.

Multiple Providers

In an oligopoly of multiple competing providers, each provider allocates resources according to the weighted proportional allocation. We assume that each provider k has a different constraint on the resources, which is captured by a convex set \mathcal{P}_k . We assume that each user can receive resources from any provider and is concerned only with the total allocation received over all providers. Note that both of these assumptions can be relaxed, as we can encode some constraints in the convex set \mathcal{P}_k . We will use the following notation. Let x_i^k denote the allocation to user i by provider k . For each user i , the utility of an allocation $(x_i^k, i = 1, \dots, n, k = 1, \dots, m)$ is $U_i(\sum_k x_i^k)$. Let $x_i = \sum_k x_i^k$ denote the total allocation to user i over all providers. We denote with $x_i^{-k} = x_i - x_i^k$ the total allocation to user i over all providers except provider k .

Let $\vec{x} = (x_i^k, i = 1, \dots, n, k = 1, \dots, m)$ be an allocation under weighted proportional sharing mechanism. It is analog to the argument in Section 5.2 that given \vec{x} , each provider k can find the weights (C_1^k, \dots, C_n^k) such that \vec{x} is the equilibrium of the weighted proportional sharing in the multiple providers' setting. In this setting the payment of user i to provider k is w_i^k , and the user's goal is to maximize $U_i(\sum_k x_i^k) - \sum_k w_i^k$, where $x_i^k = C_i^k w_i^k / \sum_i w_i^k$. The provider k , on the other hand, obtains the revenue R^k , which satisfies the following

$$\sum_{i=1}^n \frac{U'_i(x_i^{-k} + x_i^k)x_i^k}{U'_i(x_i^{-k} + x_i^k)x_i^k + R^k} = 1. \quad (5.14)$$

In order to gain some intuition, note that for every i , $U_i(x)$ is a concave function, thus $U'_i(x_i^{-k} + x_i^k)$ decreases with x_i^{-k} . From this, we can see that the marginal utility for a user with a provider k decreases if the user already received allocations from other providers. As a result, provider k may extract smaller revenue

due to competition with other providers. With this in mind, we now define an equilibrium in the case of multiple providers.

DEFINITION 5.1 *We call \vec{x} an equilibrium allocation if for every k , the allocation vector $\vec{x}^k = (x_1^k, \dots, x_n^k)$ maximizes R^k given by (5.14) over the set \mathcal{P}_k .*

We note that in the multiple providers' setting, we think of the game as the provider k 's strategy set is \mathcal{P}_k . The discrimination weights and the revenue then can be calculated according to the allocation vector \vec{x} of all providers. With these discrimination weights under users' selfish behaviour \vec{x} will be an outcome of the game. From providers' perspective, an equilibrium is an \vec{x} where no provider has an incentive to unilaterally change its allocation vector. Note that when there is only one provider, this game is the same as the two-stage Stackelberg game defined in Section 5.2.

A Class of Utility Functions

We next define a class of utility functions. The following definition may appear technical, however, it has a strong connection with the theory of third-degree price discrimination [56, 57]. It turns out that our class of utility functions covers most of interesting and commonly used utility functions in the literature. For example, the linear utilities used in sponsored search model, truncated linear utility functions or logarithmic functions considered representative of real-time traffic requirements in communication network scenarios [52], concave marginal utilities considered in [17], polynomial utility functions used in a model of trade [26], some of α -fair utility functions [39, 24] that are widely used

in the context of network resource sharing and a class of utility functions that characterize TCP-like connections [22].

DEFINITION 5.2 Let $U(x)$ be a non-negative, non-decreasing, and concave utility function and let $x_0 \geq 0$ be the value maximizing $U'(x)x$. We call $U(x)$, δ -utility, if, in addition, the following two conditions hold: (i) $U'(x)x$ is a concave function over $[0, x_0]$, and (ii) there exists $\delta \in [0, \infty)$, such that for every $a \in [0, x_0]$,

$$U(b) - [U'(a)a]'b \leq \delta U(a),$$

where b is defined as such $U'(b) = [U'(a)a]' = U'(a) + U''(a)a \geq 0$.

In Figure 5.3, we show geometric interpretations of the latter definition. In this figure, $\frac{L}{W} \leq \delta$ where (left) L and W are lengths of the line segments and (right) L is the shaded and W is the hatched area.

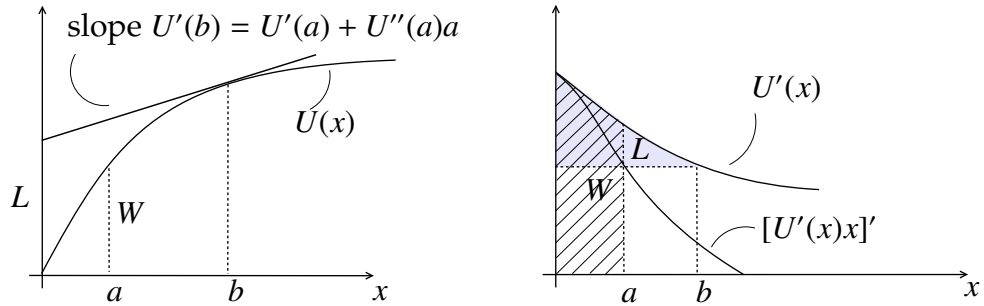


Figure 5.3: Geometric interpretations of δ -Utilities.

We have the following result.

LEMMA 5.7 If f and g are δ -utilities, then so are: $c \cdot f$, for $c > 0$ and $f + g$.

Proof. It is straightforward to show that $c \cdot f$ is a δ -utility function. In the following, we show $f + g$ is a δ -utility function

Let $h = f + g$. Given $a \geq 0$, let $b \geq 0$ be such that

$$[h'(a)a]' = h'(b). \quad (5.15)$$

We need to show that $h(b) - h'(b)b \leq \delta h(a)$, which corresponds to

$$f(b) - f'(b)b + g(b) - g'(b)b \leq \delta(f(a) + g(a)). \quad (5.16)$$

Let b_1 and b_2 be such that

$$[f'(a)a]' = f'(b_1) \quad (5.17)$$

$$[g'(a)a]' = g'(b_2) \quad (5.18)$$

and, without loss of generality, assume $b_1 \leq b_2$. Since f and g are δ -utilities, the following two relations hold

$$f(b_1) - f'(b_1)b_1 \leq \delta f(a) \text{ and } g(b_2) - g'(b_2)b_2 \leq \delta g(a).$$

Hence,

$$f(b_1) - f'(b_1)b_1 + g(b_2) - g'(b_2)b_2 \leq \delta(f(a) + g(a)). \quad (5.19)$$

In view of (5.16) and (5.19), it suffices to show that

$$f(b) - f'(b)b + g(b) - g'(b)b \leq f(b_1) - f'(b_1)b_1 + g(b_2) - g'(b_2)b_2. \quad (5.20)$$

Note that $[h'(a)a]' = [f'(a)a]' + [g'(a)a]'$. Combining with (5.15), (5.17), and (5.18), we observe

$$f'(b_1) + g'(b_2) = f'(b) + g'(b).$$

Using this identity it is not difficult to conclude that $b_1 \leq b \leq b_2$ and that we can rewrite (5.20) as

$$f(b) - f'(b_1)b + g(b) - g'(b_2)b \leq f(b_1) - f'(b_1)b_1 + g(b_2) - g'(b_2)b_2.$$

The latter inequality indeed holds if the following two inequalities hold

$$f(b) - f(b_1) \leq f'(b_1)(b - b_1)$$

$$g(b_2) - g(b) \geq g'(b_2)(b_2 - b)$$

but the latter two inequalities are indeed true as $b_1 \leq b \leq b_2$ and both f and g are concave functions. ■

Remark One can show that a linear function is 0-utility, a polynomial $U(x) = (c + x)^\alpha$ for $c \geq 0$ is a $\frac{\epsilon}{2}$ -utility for any $0 \leq \alpha \leq 1$, or a logarithmic function is a 2-utility. From this and Lemma 5.7, we can see that for example, any polynomial of the form $\sum_i a_i x^{\alpha_i}$, where $a_i > 0$ and $0 \leq \alpha_i \leq 1$ is a $\frac{\epsilon}{2}$ -utility.

The following lemma shows that many utility functions found in literature are δ -utilities.

LEMMA 5.8 *We have the following properties:*

- (i) $U(x) = \alpha x$, for $\alpha > 0$, or a truncated linear function, that is, $U(x) = \min\{\alpha x, y\}$, for every $x \geq 0$, for some $y > 0$, is a 0-utility;
- (ii) $U(x)$ such that $U'(x)$ is a concave function is a 2-utility;
- (iii) $U(x) = \log\left(\frac{c+x}{c}\right)$, for $c > 0$, is a 2-utility;
- (iv) $U(x) = (c + x)^\alpha$ for $c \geq 0$ is a $\frac{\epsilon}{2}$ -utility for $0 < \alpha \leq 1$.

Proof. We show proofs for each item in the following.

Item (i)

It suffices to consider truncated linear functions, i.e. for $\alpha > 0$ and $y > 0$, $U(x) = \min\{\alpha x, y\}$, $x \geq 0$, as linear functions are a special case with $y = \infty$. Clearly, we have $U(b) - U'(b)b = 0$, for any $b \geq 0$, hence $\delta = 0$.

Item (ii)

Consider the tangent to $U'(x)$ at the point $x = a$; see Figure 5.4. This tangent forms the triangle BDF . Note that the area L is less or equal to the area of the triangle BDF . The side DF of the triangle is of length $-2U''(a)a$. The side FB of the triangle is of length $2a$. Hence, the area of the triangle is equal to $-2U''(a)a^2$. Now, note that the area W is greater or equal to the area of the rectangle $ACEF$. The sides of this rectangle are of length $-U''(a)a$ and a . Hence, the area of the rectangle is $-U''(a)a^2$. It follows that $L/W \leq 2$.

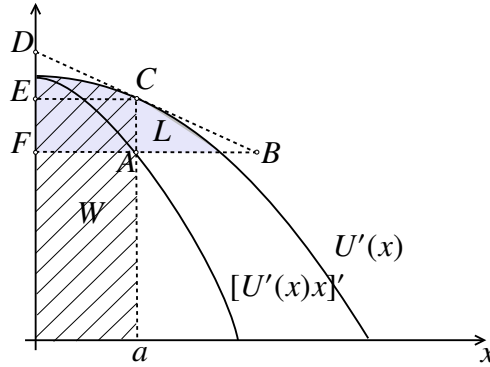


Figure 5.4: $U'(x)$ concave.

Item (iii)

We have

$$U'(x) = \frac{1}{c+x} \text{ and } [U'(x)x]' = \frac{c}{(c+x)^2}.$$

From $U'(b) = [U'(a)a]'$ we have

$$U'(b) = \frac{1}{c+b} = \frac{c}{(c+a)^2}$$

and

$$b = \frac{(c+x)^2}{c} - c.$$

It follows

$$\begin{aligned} \frac{U(b) - U'(b)b}{U(a)} &= \frac{2 \log\left(\frac{c+a}{c}\right) + \left(\frac{c}{c+a}\right)^2 - 1}{\log\left(\frac{c+a}{c}\right)} \\ &= \frac{2 \log(u) - u^2 + 1}{\log(u)} = 2 - \frac{u^2 - 1}{\log(u)} := \varphi(u) \end{aligned}$$

where $u = c/(c+a)$. Since $(u^2 - 1)/\log(u) \geq 0$, we have $\varphi(u) \leq 2$, for all $u \in [0, 1]$.

This bound is tight; achieved at $u = 0$.

Item (iv)

We have

$$\begin{aligned} U(x) &= (c+x)^\alpha \text{ and } U'(x) = \alpha(c+x)^{\alpha-1}, \\ [U'(x)x]' &= \alpha(c+x)^{\alpha-1} \left[1 - (1-\alpha)\frac{x}{c+x} \right]. \end{aligned}$$

It follows

$$\frac{U(b) - U'(b)b}{U(a)} = (1-\alpha) \left[1 - (1-\alpha)\frac{a}{c+a} \right]^{-\frac{\alpha}{1-\alpha}} + \alpha \frac{c}{c+a} \left[1 - (1-\alpha)\frac{a}{c+a} \right]. \quad (5.21)$$

Let us consider the right-hand side with the following change of variables $u = a/(c + a)$, and we denote the formula by $f_\alpha(u)$. We have

$$f_\alpha(u) := (1 - \alpha) [1 - (1 - \alpha)u]^{-\frac{\alpha}{1-\alpha}} + \alpha(1 - u) [1 - (1 - \alpha)u].$$

It is not difficult to note that $f'_\alpha(u)$ is non-decreasing on $[0, 1]$, hence $f_\alpha(u)$ is a convex function on $[0, 1]$. It follows that the function $f_\alpha(u)$ over $u \in [0, 1]$ achieves maximum at either $u = 0$ or $u = 1$, with values $f_\alpha(0) = 1$ and $f_\alpha(1) = (1 - \alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$. We have

$$f_\alpha(u) \leq \max\{1, (1 - \alpha)\alpha^{-\frac{\alpha}{1-\alpha}}\}$$

We now show that $(1 - \alpha)\alpha^{-\frac{\alpha}{1-\alpha}} \leq \frac{e}{2}$. By this we will prove the lemma.

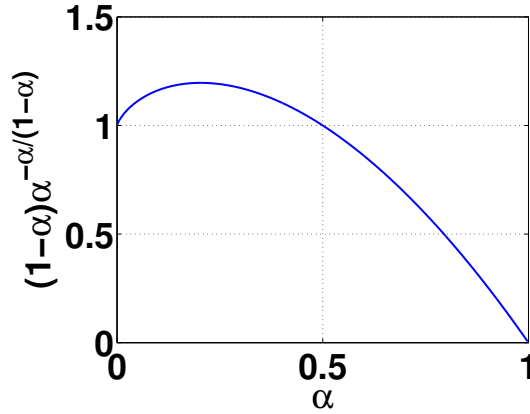


Figure 5.5: The function $(1 - \alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$ versus α .

Indeed, the function $f(\alpha) := (1 - \alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$ achieves the maximum value at the same points as the function $g(u) = \log f(\alpha)$. We have

$$g(\alpha) = \log(1 - \alpha) - \frac{\alpha}{1 - \alpha} \log(\alpha).$$

It is straightforward to obtain

$$g'(\alpha) = \frac{1}{1 - \alpha} \left[2 + \frac{1}{1 - \alpha} \log(\alpha) \right].$$

At a point α^* at which $g(\alpha)$ is maximum, we have $g'(\alpha^*) = 0$, which is equivalent to

$$\alpha^* = e^{-2(1-\alpha^*)}.$$

It follows

$$f(\alpha^*) = (1 - \alpha^*)e^{2\alpha^*} = (1 - \alpha^*)e^{-2(1-\alpha^*)}e^2 \leq e^2 \max_{x \in [0,1]} xe^{-2x} = \frac{e}{2}.$$

■

Efficiency Bound

We now state and prove the main theorem of this section.

THEOREM 5.9 *Assume that for every user i and every $a \geq 0$, $U'_i(x + a)x$ is a continuous and concave function. Then, there exists an equilibrium in the case of multiple providers defined as above. Furthermore if $U_i(a + x)$ are δ -utilities, then the efficiency at any equilibrium is at least $1/(1 + 2/\sqrt{3} + \delta)$.*

Note that when the utility functions are linear, i.e. $\delta = 0$, we have Theorem 5.4 as a special case. The result of Theorem 5.9 is rather surprising as it is not a priori clear that in a complex system where both users and providers aim at selfishly maximizing their individual payoffs (objectives which often conflict each other), the efficiency would be bounded by a constant that is independent of the number of users and the number of providers.

Proof of the first part of Theorem 5.9. The proof for the first part of the theorem about the existence of a Nash equilibrium uses standard fixed point argu-

ment. For both price taking users and price anticipating users, a Nash equilibrium is determined by a set of allocation vectors $(\vec{x}^1, \dots, \vec{x}^m) \in \mathcal{P}_1 \times \dots \times \mathcal{P}_m$. Consider the conventional best-response function

$$F : \mathcal{P}_1 \times \dots \times \mathcal{P}_m \rightarrow \mathcal{P}_1 \times \dots \times \mathcal{P}_m$$

such that $(\vec{y}^1, \dots, \vec{y}^m) = F(\vec{x}^1, \dots, \vec{x}^m)$, where \vec{y}^k is the allocation vector that maximizes the revenue for provider k , assuming other providers do not change their allocations. This mapping is continuous and thus by the fixed-point theorem, there exists an allocation vector where no provider k can increase his revenue by changing the allocation vector \vec{x}^k , which is a Nash equilibrium. ■

For the second part, the key idea of the proof is to bound the social welfare by an affine function which allows separating the maximization over $(\vec{x}^1, \dots, \vec{x}^m) \in \sum_k \mathcal{P}_k$ to maximizations over the sets \mathcal{P}_k , where $\sum_k \mathcal{P}_k := \{\vec{z}^1 + \dots + \vec{z}^m : \vec{z}^k \in \mathcal{P}_k, k = 1, \dots, m\}$ is the Minkowski sum of the sets \mathcal{P}_k . Once the optimization problem is separated, we can use a similar bound as in Section 5.4 (see Lemma 5.10 below) as a subroutine to prove the theorem. Now, let

$$v_i^k = U_i'(x_i) + U_i''(x_i)x_i^k, \text{ for each } i, \text{ and } v_i = \min_k v_i^k.$$

Since for every i , $U_i(x)$ is a concave function, $U_i''(x)$ is non-positive, and thus, $v_i = U_i'(x_i) + U_i''(x_i)(\max_k x_i^k) \geq U_i'(x_i) + U_i''(x_i)x_i$. The last inequality is because of the fact $x_i = \sum_k x_i^k$.

Now, let us define $V_i(x) = a_i + v_i x$ where a_i is chosen so that $V_i(x)$ is a tangent to $U_i(x)$. We will use V_i as an upper bound of U_i . By the definition of δ -utility functions, we have $a_i \leq L \leq \delta U_i(x_i)$, where

$$L = U_i(x_i) - (U_i'(x_i) + U_i''(x_i)x_i)y_i$$

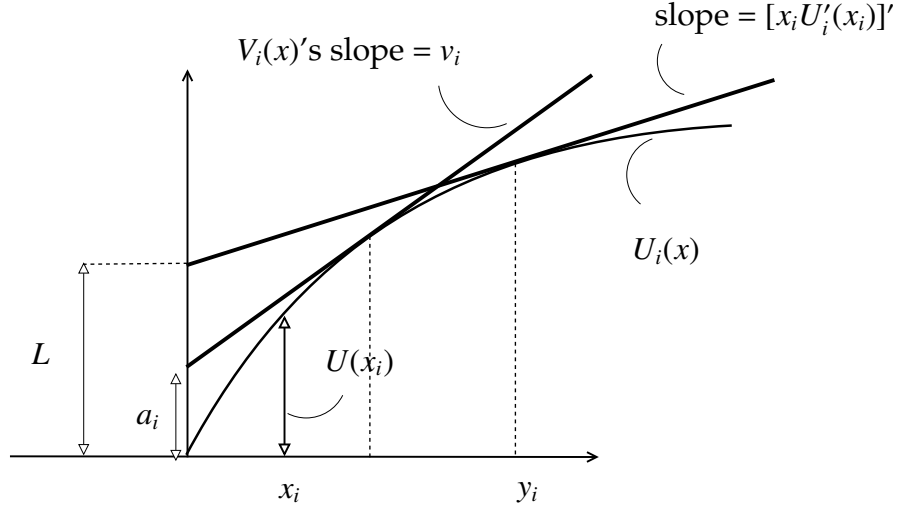


Figure 5.6: Upper-bounding $U_i(x)$ by an affine function $V_i(x)$.

and y_i is defined such that $U'_i(y_i) = U'_i(x_i) + U''_i(x_i)x_i$. (See Figure 5.6.) Therefore,

$$\sum_i a_i \leq \delta \sum U_i(x_i). \quad (5.22)$$

Since $U_i(x)$ is a non-negative concave function, we have $U_i(x) \leq V_i(x)$. Hence,

$$\begin{aligned} \max_{z \in \sum_k \mathcal{P}_k} \sum_i U_i(z_i) &\leq \max_{z \in \sum_k \mathcal{P}_k} \sum_i V_i(z_i) = \\ &= \sum_i a_i + \max_{z \in \sum_k \mathcal{P}_k} \sum_i v_i z_i = \sum_i a_i + \sum_k \max_{z \in \mathcal{P}_k} \sum_i v_i z_i. \end{aligned} \quad (5.23)$$

The last is a key inequality as it enables us to use the fact that $v_i z_i$ are linear functions, therefore, instead of considering the maximization over the set $\sum_k \mathcal{P}_k$ we can bound $\sum_i v_i z_i$ over each \mathcal{P}_k .

By similar arguments as in the proof of Theorem 5.4, we can prove the following lemma.

LEMMA 5.10 *For ever k ,*

$$\sum_i U'_i(x_i) x_i^k \geq \frac{1}{1 + 2/\sqrt{3}} \max_{z \in \mathcal{P}_k} \sum_i v_i^k z_i. \quad (5.24)$$

Proof. By similar argument as in the proof of Theorem 5.4, we can assume the convex set \mathcal{P}_k is of the form

$$\sum_i \gamma_i x_i^k = 1, \text{ with } \gamma_i \geq 0 \text{ for each } i,$$

and we can derive the condition

$$\text{either } x_i = 0 \text{ or } \frac{\frac{1}{\gamma_i} [U'_i(x_i) x_i^k]'}{(U'_i(x_i) x_i^k + R^k)^2} = \frac{\frac{1}{\gamma_i} v_i^k}{(U'_i(x_i) x_i^k + R^k)^2} = p > 0. \quad (5.25)$$

Let us use the following notation

$$a_i = \frac{v_i^k}{\gamma_i} \text{ and } y_i = \frac{U'_i(x_i) x_i^k}{U'_i(x_i) x_i^k + R^k} \text{ for each user } i. \quad (5.26)$$

Without loss of generality, assume that $a_1 \geq a_2 \geq \dots \geq a_n$.

From (5.26), we have

$$\begin{aligned} \sum_i U'_i(x_i) x_i^k &= R^k \sum_i \frac{y_i}{1 - y_i} \geq R^k \left(\frac{y_1}{1 - y_1} + \sum_{i \geq 2} y_i \right) = \\ &= R^k \left(\frac{y_1}{1 - y_1} + (1 - y_1) \right) = R^k \frac{y_1^2 - y_1 + 1}{1 - y_1}. \end{aligned} \quad (5.27)$$

While

$$\max_{z \in \mathcal{P}_k} \sum_i v_i^k z_i = \max_i a_i = a_1.$$

It is straightforward to see that the following holds $U'_i(x_i) x_i^k = R^k \frac{y_i}{1 - y_i}$.

Therefore, for every i ,

$$\gamma_i x_i^k = R^k \frac{\gamma_i}{U'_i(x_i)} \frac{y_i}{1 - y_i}.$$

However,

$$a_i = \frac{[U'_i(x_i) x_i^k]'}{\gamma_i} \leq \frac{U'_i(x_i)}{\gamma_i}.$$

Therefore, we have

$$\gamma_i x_i^k = R^k \frac{\gamma_i}{U'_i(x_i)} \frac{y_i}{1 - y_i} \leq R^k \frac{1}{a_i} \frac{y_i}{1 - y_i}.$$

Thus,

$$a_1 = a_1 \left(\sum_i z_i \right) \gamma_i x_i^k \leq a_1 R^* \sum_i \frac{y_i}{a_i(1-y_i)}$$

We also have

$$\text{either } y_i = 0 \text{ or } \frac{a_i(1-y_i)^2}{(R^k)^2} = p > 0. \quad (5.28)$$

From the latter, the analysis follows the same steps as in the proof of Theorem 5.4, which yields the result

$$\sum_i U'_i(x_i) x_i^k \geq \frac{1}{1 + 2/\sqrt{3}} \max_{z \in \mathcal{P}_k} \sum_i v_i^k z_i.$$

■.

We now use this Lemma to prove our main result.

Proof of the second part of Theorem 5.9 On the one hand, if we sum the left-hand side of (5.24) over all k , we have

$$\sum_{k,i} U'_i(x_i) x_i^k = \sum_i U'_i(x_i) x_i \leq \sum_i U_i(x_i) \quad (5.29)$$

where the last inequality is true because $U_i(x)$ is a non-negative and concave function for every i .

On the other hand, if we sum the right-hand side of (5.24) over all k , we obtain

$$\sum_k \frac{1}{1 + 2/\sqrt{3}} \max_{z \in \mathcal{P}_k} \sum_i v_i^k z_i \geq \frac{1}{1 + 2/\sqrt{3}} \sum_k \max_{z \in \mathcal{P}_k} \sum_i v_i z_i, \quad (5.30)$$

where in the last inequality, v_i^k are replaced by v_i , which recall is equal to $\min_k v_i^k$.

Combining (5.24), (5.29) and (5.30), we derive

$$\sum_i U_i(x_i) \geq \frac{1}{1 + 2/\sqrt{3}} \sum_k \max_{z \in \mathcal{P}_k} \sum_i v_i z_i.$$

From this we obtain

$$(1 + 2/\sqrt{3}) \sum_i U_i(x_i) \geq \sum_k \max_{z \in \mathcal{P}_k} \sum_i v_i z_i. \quad (5.31)$$

Finally, from (5.22), (5.23) and (5.31), we have

$$\max_{z \in \sum_k \mathcal{P}_k} \sum_i U_i(z_i) \leq (\delta + 1 + 2/\sqrt{3}) \sum_i U_i(x_i),$$

which establishes the asserted result. ■

5.6 Related Literature

Our results in this chapter follows the line of work on proportional sharing of [23, 20, 42]. Most of these results, however, do not investigate the incentive of profit maximizing providers. There are some recent works investigating the revenue of more general proportional mechanisms which is called quasi-proportional sharing [38, 44]. The work presented in this chapter is based on a joint work with M. Vojnović [43].

The literature on sponsored search applications is expanding in both economics [58, 10, 6] and computer science [3, 36]. Also see the survey [28] and the references therein.

The results in this chapter also analyze the case of multiple providers competing for the same set of buyers. Similar problems are considered in [1, 2, 17].

CHAPTER 6

CONCLUSION

6.1 Summary

In this thesis, we studied the classical proportional sharing mechanism for general polyhedral environment with the focus on the revenue of the mechanism. The proportional sharing is a natural, simple and robust mechanism. This is the main motivation to investigate this class of mechanisms. The insights we learned from this investigation are:

- Proportional sharing is not only efficient, natural and scalable but also generates high revenue under a symmetric condition among users.
- Without the symmetric condition, the proportional sharing mechanism can be combined with other mechanisms to gain competitive revenue.
- The weighted proportional sharing can be studied as a framework where providers aim to maximize the revenue under a discrimination pricing scheme. This is natural and quite efficient even in a complex environment of multiple providers competing for profit.
- For the application of keyword auctions, we formulate a general model for the keyword auction application. Our approach can model a complex externalities among advertisers. We believe this is an important feature that needs further investigation and the model can be used for other applications as well.

6.2 Future Research

There are several open directions for future research. I will describe some general directions that I think are important and promising.

Nash Implementation versus Truthful Mechanisms The thesis uses Nash equilibrium as the solution concept in designing mechanism. Nash implementation in full information settings assumes that users know about all others' information, but the planner does *not* have this knowledge. Nash implementation is a strictly weaker concept than truthful mechanisms. However, the following question is not well understood. How much more revenue can we gain by using Nash implementation rather than using truthful mechanisms?

Collusion in Mechanism Design In computer science applications, the condition on simplicity and robustness of mechanisms are crucial. These issues are the current challenges of the area. One of the well recognized approaches is the prior-free mechanism design. That is, to design mechanism without the knowledge of the types's distributions. But there are other directions concerning the robustness of mechanisms as well. For example, the robustness conditions against the collusion of users. Several models have been proposed and studied in Bayesian settings, such as the ring model of McAfee and McMillan [35] or the mechanism of Laffont [27].

It is of great interests to investigate the effects on the revenue of collusion in mechanism design in prior-free settings or in Nash equilibria of full information games. Allowing bidders to collude in an arbitrary way can lead to a very low revenue. One of the possible directions is to consider the collusion structures

that can be represented by a set system: only bidders that belong to the same set can collude.

Mechanism Design with Many Sellers One of the main criticism in the field of mechanism design is about the *control assumption*, as named by T. Palfrey [47], which stipulates that the planner can control all communication that agents can undertake. When this control is not feasible, the system is ruled by complex interaction among multiple mechanism designers. In fact, this is the case in most systems of computer science settings. Research in Bayesian multi-contracting mechanism design is still in early stage, see the survey of D. Martinmort [31] and recent works of M. Pai [45, 46]. The difficulty in this area is that the revelation principle does not hold when there are many sellers. The direct revelation mechanisms generally do not suffice to describe the whole set of equilibrium allocation. And in general, analyzing multi-sellers buyers games is fairly complex in Bayesian settings. It is of great interests to define special cases of these Bayesian games, where there is a characterization as in the case of single seller [40]. On the other hand, the situation is easier in full information settings, see for example the work of Blume et al. [9], one of results in this thesis also extends to the case of multiple providers.

A hybrid approach between Bayesian and full information games might lead to interesting results. Can we model and analyze games consisting of multiple players that capture some types of information asymmetry among players? This is a challenging problem that goes beyond the well understood model of mechanism design with single seller and will have impact on other related research areas

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